Propagation of a Gaussian wave packet

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We study the time evolution of an initially Gaussian pulse.

1 Gaussian integrals

Before starting our example, we show how to find the integral of a Gaussian curve,

$$I = \int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/2\sigma^2}$$

First, change the integration variable to

$$\xi = \frac{x - x_0}{\sqrt{2}\sigma}$$

so that

$$I = \sqrt{2\sigma^2} \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2}$$

Now square I,

$$\begin{aligned} I^2 &= 2\sigma^2 \int_{-\infty}^{\infty} d\xi_1\, e^{-\xi_1^2} \int_{-\infty}^{\infty} d\xi_2\, e^{-\xi_2^2} \\ &= 2\sigma^2 \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2\, e^{-\left(\xi_1^2+\xi_2^2\right)} \end{aligned}$$

and change to polar coordinates, where $\rho^2 = \xi_1^2 + \xi_2^2$ and $d\xi_1 d\xi_2 = \rho d\rho d\varphi$. Then

$$\begin{aligned} I^2 &= 2\sigma^2 \int_0^{2\pi} d\varphi \int_0^{\infty} \rho d\rho\, e^{-\rho^2} \\ &= 4\pi\sigma^2 \int_0^{\infty} \rho d\rho\, e^{-\rho^2} \end{aligned}$$

and another change of variable to $\lambda = \rho^2$ gives a simple exponential, with $\int_0^\infty d\lambda\, e^{-\lambda} = 1$,

$$\begin{aligned} I^2 &= 2\pi\sigma^2 \int_0^{\infty} d\lambda\, e^{-\lambda} \\ &= 2\pi\sigma^2 \end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} dx\, e^{-(x-x_0)^2/2\sigma^2} = \sqrt{2\pi\sigma^2}$$

2 Initial conditions and the mode amplitude

We begin with an initially Gaussian pulse

$$u(x,0) = e^{-x^2/2L^2} e^{ik_0 x}$$

with zero initial rate of change,

$$\frac{\partial u}{\partial t}(x,0) = 0$$

We use these initial data to find $A(k)$, then use $A(k)$ to find $u(x,t)$.

The solution for $A(k)$ is found by integrating over the initial conditions

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left(u(x,0) - \frac{i}{\omega(k)} \frac{\partial u}{\partial t}(x,0) \right) e^{-ikx}$$

Substituting for $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$ gives a Gaussian integral,

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2L^2} e^{ik_0 x} e^{-ikx}$$

For the integral of a Gaussian, see the final section. integral, we add and subtract a constant to complete the square in the exponent,

$$\begin{aligned} -\frac{1}{2L^2}\left(x^2 - 2iL^2(k-k_0)x\right) &= -\frac{1}{2L^2}\left[\left(x^2 - 2iL^2(k-k_0)x + \left(iL^2(k-k_0)\right)^2\right) - \left(iL^2(k-k_0)\right)^2\right] \\ &= -\frac{1}{2L^2}\left(x - iL^2(k-k_0)\right)^2 - \frac{1}{2}L^2(k-k_0)^2 \end{aligned}$$

Then, performing the Gaussian integral,

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{-x^2/2L^2} e^{-ikx} e^{ik_0 x} &= \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2L^2}\left(x - iL^2(k-k_0)\right)^2 - \frac{1}{2}L^2(k-k_0)^2} \\ &= e^{-\frac{1}{2}L^2(k-k_0)^2} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2L^2}\left(x - iL^2(k-k_0)\right)^2} \\ &= \sqrt{2\pi L^2} e^{-\frac{1}{2}L^2(k-k_0)^2} \end{aligned}$$

Therefore, the mode amplitudes follow a Gaussian as well,

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}}\left(\sqrt{2\pi} L e^{\frac{1}{2L^2}\left[iL^2(k-k_0)\right]^2}\right) \\ &= L e^{-\frac{L^2}{2}(k-k_0)^2} \end{aligned}$$

3 Dispersion relation

The time dependence of the wave packet is now given by

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\left[A(k)\,e^{i(kx-\omega t)} + A^*(k)\,e^{-i(kx-\omega t)}\right] \\
&= \frac{1}{2}\frac{L}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2}(k-k_0)^2}\left(e^{i(kx-\omega t)} + e^{-i(kx-\omega t)}\right) \\
&= \frac{L}{\sqrt{2\pi}}\mathcal{R}e\int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2}(k-k_0)^2} e^{i(kx-\omega t)}
\end{aligned}$$

where $\omega = \omega(k)$.

Therefor, to continue, we need the form of of the dispersion relation, $\omega(k)$. Consider the case of high-frequency waves in plasma, for which we found

$$\omega = \sqrt{k^2c^2 + \omega_p^2}$$

Now we let $\omega_P \gg kc$ so that we may expand the square root,

$$\begin{aligned}
\omega &\approx \omega_P\left(1 + \frac{c^2}{2\omega_P^2}k^2\right) \\
&\approx \omega_P\left(1 + \frac{1}{2}l_P^2k^2\right)
\end{aligned}$$

where we define the plasma wavelength, $l_P = \frac{c}{\omega_P}$. This gives our dispersion relation. The group velocity is

$$v_g = \frac{d\omega}{dk} = \omega_P l_P^2 k$$

so the group velocity evaluated at the center of the packet is

$$v_0 \equiv \omega_P l_P^2 k_0$$

and we may write

$$\omega \approx \omega_P + \frac{1}{2}kv_g$$

Notice that the approximation means that

$$\begin{aligned}
\frac{v_g}{c} &= \frac{1}{2c}l_P^2\omega_P k \\
&= \frac{1}{2\omega_P}kc \\
&\ll 1
\end{aligned}$$

so the group velocity is much less than the speed of light.

Notice that this form also works for a Klein-Gordon matter wave when the momentum due to mass is much greater than the wave momentum, $mc \gg \hbar k$. Expanding,

$$\omega = \sqrt{\frac{m^2c^4}{\hbar^2} + k^2c^2}$$

$$
\begin{aligned}
&= \frac{mc^2}{\hbar}\sqrt{1+\frac{\hbar^2k^2c^2}{m^2c^4}} \\
&\approx \frac{mc^2}{\hbar}\left(1+\frac{1}{2}\frac{\hbar^2}{m^2c^2}k^2\right) \\
&= \omega_C\left(1+\frac{1}{2}\lambda^2k^2\right) \\
&= \omega_C+\frac{1}{2}kv_g
\end{aligned}
$$

where $\lambda_C = \frac{\hbar}{mc}$ is the Compton wavelength and $\omega_C = \frac{c}{\lambda_C}$ the corresponding frequency. We continue below with a wave packet in plasma.

4 Time evolution of the waveform

With this dispersion relation, $\omega = \omega_P + \frac{1}{2}kv_g$, the full time evolution of the wave is given by

$$
u(x,t) = \frac{L}{\sqrt{2\pi}}\mathcal{R}e\int_{-\infty}^{\infty} dk e^{-\frac{L^2}{2}(k-k_0)^2} e^{i\left(kx-\omega_P+\frac{1}{2}kv_gt\right)}
$$

Substitinting for ω we have another Gaussian integral,

$$
\begin{aligned}
u(x,t) &= \frac{L}{\sqrt{2\pi}}\mathcal{R}e\int_{-\infty}^{\infty} dk\, e^{-\frac{L^2}{2}(k-k_0)^2} e^{i\left(kx-\left(\omega_P\left(1+\frac{1}{2}l_P^2k^2\right)\right)t\right)} \\
&= \frac{L}{\sqrt{2\pi}}Re\int_{-\infty}^{\infty} dk\, \exp\left[-\frac{L^2}{2}(k-k_0)^2+ikx-i\omega_Pt-\frac{i}{2}\omega_Pl_P^2k^2t\right]
\end{aligned}
$$

Expanding the exponent,

$$
\begin{aligned}
-\frac{L^2}{2}(k-k_0)^2+ikx-i\omega_Pt-\frac{i}{2}\omega_Pl_P^2k^2t &= -\frac{L^2}{2}\left((k-k_0)^2-\frac{2}{L^2}ikx+\frac{2}{L^2}i\omega_Pt+\frac{i}{2}\frac{2}{L^2}\omega_Pl_P^2k^2t\right) \\
&= -\frac{L^2}{2}\left(k^2-2kk_0+k_0^2-\frac{2ix}{L^2}k+\frac{2}{L^2}i\omega_Pt+\frac{i\omega_Pl_P^2t}{L^2}k^2\right) \\
&= -\frac{L^2}{2}\left(\left(1+\frac{i\omega_Pl_P^2t}{L^2}\right)k^2-2\left(k_0+\frac{ix}{L^2}\right)k\right)-\frac{L^2}{2}\left(k_0^2+\frac{2}{L^2}i\omega_Pt\right)
\end{aligned}
$$

we complete the square,

$$
\begin{aligned}
\left(1+\frac{i\omega_Pl_P^2t}{L^2}\right)k^2-2\left(k_0+\frac{ix}{L^2}\right)k &= \left(k\sqrt{1+\frac{i\omega_Pl_P^2t}{L^2}}-\frac{k_0+\frac{ix}{L^2}}{\sqrt{1+\frac{i\omega_Pl_P^2t}{L^2}}}\right)^2-\left(\frac{k_0+\frac{ix}{L^2}}{\sqrt{1+\frac{i\omega_Pl_P^2t}{L^2}}}\right)^2 \\
&= \left(1+\frac{i\omega_Pl_P^2t}{L^2}\right)\left(k-\frac{k_0+\frac{ix}{L^2}}{1+\frac{i\omega_Pl_P^2t}{L^2}}\right)^2-\frac{\left(k_0+\frac{ix}{L^2}\right)^2}{1+\frac{i\omega_Pl_P^2t}{L^2}}
\end{aligned}
$$

Two constant exponentials,

$$
\exp\left[-\frac{L^2}{2}\left(k_0^2+\frac{2}{L^2}i\omega_Pt\right)+\frac{L^2}{2}\frac{\left(k_0+\frac{ix}{L^2}\right)^2}{1+\frac{i\omega_Pl_P^2t}{L^2}}\right] = \exp\left[-\frac{1}{2}\left(k_0^2L^2+2i\omega_Pt\right)+\frac{1}{2}\frac{\left(k_0L^2+ix\right)^2}{L^2+i\omega_Pl_P^2t}\right]
$$

come out of the integral, while the remaining Gaussian integral gives

$$\int_{-\infty}^{\infty} dk \, \exp\left(-\frac{1}{2}\left(L^2 + i\omega_P l_P^2 t\right)\left(k - \frac{k_0 + \frac{ix}{L^2}}{1 + \frac{i\omega_P l_P^2 t}{L^2}}\right)^2\right) = \frac{\sqrt{2\pi}}{\sqrt{L^2 + i\omega_P l_P^2 t}}$$

Combining these,

$$u(x,t) = \mathcal{R}e \frac{L}{\sqrt{L^2 + i l_P ct}} \exp\left[-\frac{1}{2}\left(k_0^2 L^2 + 2i\omega_P t\right) + \frac{1}{2}\frac{\left(k_0 L^2 + ix\right)^2}{L^2 + i l_P ct}\right]$$

where we have used $l_P \omega_P = c$. We easily check that at $t = 0$ we recover the initial state,

$$\begin{aligned} u(x,0) &= \mathcal{R}e \exp\left[-\frac{1}{2}\left(k_0^2 L^2\right) + \frac{1}{2}\frac{\left(k_0 L^2 + ix\right)^2}{L^2}\right] \\ &= \mathcal{R}e \exp\left[-\frac{1}{2}\left(k_0^2 L^2\right) + \frac{1}{2}\frac{\left(k_0 L^2 + ix\right)^2}{L^2}\right] \\ &= e^{-x^2/2L^2} e^{ik_0 x} \end{aligned}$$

5 Finding the real part

Finally, we find the real part of the waveform. Separating the real and imaginary parts of the argument of the exponential,

$$\begin{aligned} -\frac{1}{2}\left(k_0^2 L^2 + 2i\omega_P t\right) + \frac{1}{2}\frac{\left(k_0 L^2 + ix\right)^2}{L^2 + i l_P ct} &= -\frac{1}{2}\left(k_0^2 L^2 + 2i\omega_P t\right) + \frac{1}{2}\frac{\left(L^2 - i l_P ct\right)\left(k_0^2 L^4 + 2ixk_0 L^2 - x^2\right)}{L^4 + l_P^2 c^2 t^2} \\ &= -\frac{1}{2}k_0^2 L^2 + \frac{1}{2}\frac{k_0^2 L^4 L^2 + 2xk_0 L^2 l_P ct - x^2 L^2}{L^4 + l_P^2 c^2 t^2} \\ &\quad -i\omega_P t + \frac{i}{2}\frac{-2xk_0 L^4 + l_P x^2 ct + k_0^2 L^4 l_P ct}{L^4 + l_P^2 c^2 t^2} \end{aligned}$$

The real part of this may be written as

$$\begin{aligned} -\frac{1}{2}k_0^2 L^2 + \frac{1}{2}\frac{k_0^2 L^4 L^2 + 2xk_0 L^2 l_P ct - x^2 L^2}{L^4 + l_P^2 c^2 t^2} &= \frac{1}{2}\frac{-k_0^2 L^2 l_P^2 c^2 t^2 + 2xk_0 L^2 l_P ct - x^2 L^2}{L^4 + l_P^2 c^2 t^2} \\ &= -\frac{1}{2}\frac{L^2}{L^4 + l_P^2 c^2 t^2}\left(x - k_0 l_P ct\right)^2 \end{aligned}$$

Recalling that the group velocity, evaluated at the central wave number, is $v_0 = k_0 l_p c$, the real part of this has the form of a Gaussian with a time-dependent mean,

$$\exp\left(-\frac{\left(x - v_0 t\right)^2}{2\sigma^2(t)}\right)$$

where

$$\sigma(t) = \sqrt{L^2 + \frac{l_P^2}{L^2}c^2 t^2}$$

For the imaginary part, consider the limit as t becomes large, $l_P ct \gg L^2$ near the center of the Gaussian, $x \approx v_0 t$,

$$\begin{aligned}
-i\omega_P t + \frac{i}{2}\frac{-2xk_0L^4 + l_P x^2 ct + k_0^2 L^4 l_P ct}{L^4 + l_P^2 c^2 t^2} &\rightarrow -i\omega_P t + \frac{i}{2}\frac{k_0 L^4 \left(-2x + v_0 t\right) + x^2 l_P ct}{L^4 + l_P^2 c^2 t^2} \\
&\approx -i\left(1 - \frac{1}{2}\frac{v_0^2}{c^2}\right)\omega_P t \\
&\approx -i\omega_P t
\end{aligned}$$

since the group velocity is much less than c.

Finally, the amplitude of the exponential is

$$\begin{aligned}
\frac{L}{\sqrt{L^2 + il_P ct}} &= \frac{1}{\sqrt{1 + i\frac{l_P ct}{L^2}}} \\
&= \frac{1}{\sqrt{1 + \frac{l_P^2 c^2 t^2}{L^4}} e^{i\tan^{-1}\frac{l_P ct}{L^2}}} \\
&= \frac{1}{\sqrt{1 + \frac{l_P^2 c^2 t^2}{L^4}}} e^{i\tan^{-1}\frac{l_P ct}{L^2}}
\end{aligned}$$

so at large t,

$$\frac{L}{\sqrt{L^2 + il_P ct}} \approx \frac{L^2}{l_p ct} e^{\frac{i\pi}{2}}$$

Combining the results for large t,

$$\begin{aligned}
u\left(x,t\right) &= \mathcal{R}e\left(\frac{L^2}{l_p ct} e^{\frac{i\pi}{2}} e^{-i\omega_P t}\right)\exp\left(-\frac{\left(x - v_0 t\right)^2}{2\sigma^2\left(t\right)}\right) \\
&= \frac{L^2}{l_p ct}\exp\left(-\frac{\left(x - v_0 t\right)^2}{2\sigma^2\left(t\right)}\right)\sin\omega_P t
\end{aligned}$$

The peak of the wave moves with velocity v_0 to the right, while the amplitude decreases linearly with time. The whole oscillates with frequency ω_P.