Superposition

February 12, 2016

So far we have looked at properties of monochromatic plane waves. A more complete picture is found by looking at superpositions of many frequencies. Many of the important features emerge by considering the one-dimensional case, with each mode described by

$$u(x,t) = Ae^{i(kx-\omega t)}$$

If we assume that $u(x,t)$ satisfies some form of second order wave equation (though not necessarily $\Box u = 0$), then we expect some necessary relationship – a *dispersion relation* – of the form

$$\omega = \omega(k)$$

in order to solve the wave equation. We have seen examples of this. For our damped, driven model of an electromagnetic wave in a material, we found that at high frequencies,

$$\omega = \sqrt{k^2c^2 + \omega_p^2}$$

where $\omega_p \equiv \sqrt{\frac{NZe^2}{m\epsilon_0}}$ is the plasma frequency, while for a wave traveling in a constant magnetic field we have a more complicated relationship between ω and k,

$$\begin{aligned}
kc &= \omega\sqrt{1 - \frac{\omega_P^2}{\omega(\omega \mp \omega_B)}} \\
k^2c^2(\omega \mp \omega_B) &= \omega^2(\omega \mp \omega_B) - \omega\omega_P^2 \\
0 &= \omega^3 \mp \omega^2\omega_B - \omega\left(\omega_P^2 + k^2c^2\right) \pm k^2c^2\omega_B
\end{aligned}$$

where $\omega_B = \frac{eB_0}{m}$. In this case $\omega(k)$ is the solution to a cubic equation.

The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

For a constant potential, V_0, we have plane wave solutions, $\psi(\mathbf{x},t) = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$,

$$\frac{\hbar^2\mathbf{k}^2}{2m}Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + V_0Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \hbar\omega Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

with dispersion relation

$$\omega = \frac{1}{\hbar}V_0 + \frac{\hbar\mathbf{k}^2}{2m}$$

We will assume a general relationship for the dispersion relation, requiring only that $\omega(k)$ may be expanded in a Taylor series.

1 General superposition

A superposition of plane waves may be accomplished by integrating over a range of different wavelengths and frequencies. The amplitude, A, may be different for the different modes, so the general superposition that still satisfies the wave equation will have the form

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega A(k,\omega) e^{i(kx-\omega t)} \delta(\omega - \omega(k))$$

where the delta function insures that the wave equation is satisfied. Integrating over frequency, we have

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k,\omega(k)) e^{i(kx-\omega(k)t)}$$

With the understanding that $\omega = \omega(k)$ and $A(k) = A(k,\omega(k))$, we write more concisely

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{i(kx-\omega t)}$$

1.1 Initial conditions

1.1.1 First order wave equations

The solution above is correct if the wave equation is linear in time derivatives like the Schrödinger equation. Then we require only the single initial condition

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k) e^{ikx}$$

to find the value of $A(k)$ by inverting the Fourier transform,

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, u(x,0) e^{-ikx}$$

and thereby predict the full time evolution of the wave,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx'\, u(x',0) e^{-ikx'} \right) e^{i(kx-\omega t)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx'\, u(x',0) e^{-ikx'} e^{i(kx-\omega t)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, u(x',0) \int_{-\infty}^{\infty} dk\, e^{i\left(k\left(x-x'\right)-\omega(k)t\right)} \end{aligned}$$

1.1.2 Second order wave equations

More frequently, the wave equation is second order in time. Then the condition $\omega = \omega(k)$ will be the solution to a quadratic equation, and we will get both positive and negative solutions for the frequency. In such cases, the solution includes two terms, which may be written as a complex amplitude and its conjugate,

$$u(x,t) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\left[A(k)\,e^{ikx}e^{-i\omega t} + A^*(k)\,e^{-ikx}e^{i\omega t}\right]$$

and this gives enough freedom to satisfy the two initial conditions, $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$:

$$\begin{aligned}
u(x,0) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\left[A(k)\,e^{ikx} + A^*(k)\,e^{-ikx}\right] \\
\frac{\partial u}{\partial t}(x,0) &= -\frac{1}{2}\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \omega dk\left[A(k)\,e^{ikx} - A^*(k)\,e^{-ikx}\right]
\end{aligned}$$

Inverting these Fourier transforms gives the real and imaginary parts of $A(k)$ in terms of the initial conditions,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\,u(x,0)\,e^{-ik'x} &= \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dk\left[A(k)\,e^{ikx}e^{-ik'x} + A^*(k)\,e^{-ikx}e^{-ik'x}\right] \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left[A(k)\,2\pi\delta(k-k') + A^*(k)\,2\pi\delta(k+k')\right] \\
&= \frac{1}{2}\left(A(k') + A^*(-k')\right) \\
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\frac{\partial u}{\partial t}(x,0)\,e^{-ik'x} &= -\frac{1}{2}\frac{i}{2\pi}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} \omega(k)\,dk\left[A(k)\,e^{i(k-k')x} - A^*(k)\,e^{-i(k+k')x}\right] \\
&= -\frac{i}{2}\int_{-\infty}^{\infty} \omega(k)\,dk\left[A(k)\,\delta(k-k') - A^*(k)\,\delta(k+k')\right] \\
&= -\frac{i}{2}\omega(k')\left(A(k') - A^*(-k')\right)
\end{aligned}$$

Solving for the sum and difference,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\,u(x,0)\,e^{-ik'x} &= \frac{1}{2}\left(A(k') + A^*(-k')\right) \\
\frac{i}{\sqrt{2\pi}\omega(k')}\int_{-\infty}^{\infty} dx\frac{\partial u}{\partial t}(x,0)\,e^{-ik'x} &= \frac{1}{2}\left(A(k') - A^*(-k')\right)
\end{aligned}$$

so adding gives $A(k')$. Dropping the primes,

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\left(u(x,0) + \frac{i}{\omega(k)}\frac{\partial u}{\partial t}(x,0)\right)e^{-ikx}$$

so the complex $A(k)$ is fully determined by the pair of initial conditions. Substituting this back into $u(x,t)$ gives the full time evolution.

2 Phase and group velocity

While the phase of a single wave mode propagates with velocity $v_p = \frac{\omega}{k}$, a superposition over many frequencies behaves differently.

For simplicity consider the first order solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k)\, e^{i(kx-\omega t)}$$

where we take $A(k)$ to be a smooth distribution of frequencies peaked around some value k_0. Such a superposition is sometimes called a wave packet. For this sort of superposition, we can expand the frequency in a Taylor series as

$$\omega(k) = \omega_0 + \frac{\partial \omega}{\partial k}(k-k_0) + \ldots$$

We compute $u(x,t)$ to this linear order.

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k)\, e^{i(kx-\omega t)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k)\, e^{i\left(kx-\left(\omega_0 t + \frac{\partial \omega}{\partial k}(k-k_0)t\right)\right)} \\
&= \frac{1}{\sqrt{2\pi}} e^{i\left(k_0 \frac{\partial \omega}{\partial k} - \omega_0\right)t} \int_{-\infty}^{\infty} dk A(k)\, e^{ik\left(x - \frac{\partial \omega}{\partial k} t\right)} \\
&= \frac{1}{\sqrt{2\pi}} e^{i\tilde{\omega}_0 t} \int_{-\infty}^{\infty} dk A(k)\, e^{ik\left(x - \frac{\partial \omega}{\partial k} t\right)}
\end{aligned}$$

Let $x' = x - \frac{\partial \omega}{\partial k} t$. Then

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k)\, e^{i(kx-\omega t)} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k)\, e^{ikx'} \\
&= u(x',0)
\end{aligned}$$

so the solution is

$$\begin{aligned}
u(x,t) &= e^{i\tilde{\omega}_0 t} u(x',0) \\
&= e^{i\tilde{\omega}_0 t} u\left(x - \frac{\partial \omega}{\partial k} t, 0\right)
\end{aligned}$$

and aside from an overall phase, the waveform at time t is given by the initial waveform displaced by $\frac{\partial \omega}{\partial k} t$. Thus, the wave packet moves to the right with velocity

$$v_g = \frac{\partial \omega}{\partial k}$$

This is called the *group velocity*.

2.1 Group velocity in a uniform linear medium

For light waves in a uniform medium, we know that the wave number and frequency are related by $k = \sqrt{\mu\varepsilon}\omega$, so the phase velocity is

$$\begin{aligned} v_{phase} &= \frac{\omega}{k} \\ &= \frac{1}{\sqrt{\mu\varepsilon}} \\ &= \frac{c}{n} \end{aligned}$$

If the dielectric constant (and hence the index of refraction) is independent of frequency, then with $\omega = \frac{kc}{n}$ the group velocity is the same as the phase velocity

$$v_{group} = \frac{\partial\omega}{\partial k} = \frac{c}{n}$$

This is always the case in free space, with $v_{ph} = v_g = c$.

2.2 Frequency dependence of the index of refraction

As we have seen, in many materials the index of refraction, $n = \sqrt{\mu\epsilon}$ depends on frequency. With $\omega(k)$, we may write the index of refraction as a function of wave number, $n = n(\omega(k))$. Starting from the differential of ω,

$$\begin{aligned} \omega &= \frac{kc}{n(k)} \\ d\omega &= \frac{c}{n(k)}dk - \frac{kc}{n^2}\frac{dn}{d\omega}\frac{d\omega}{dk}dk \end{aligned}$$

we write the group velocity

$$\begin{aligned} v_g &= \frac{d\omega}{dk} \\ &= \frac{c}{n(k)} - \frac{kc}{n^2}\frac{dn}{d\omega}\frac{d\omega}{dk} \\ \left(1 + \frac{kc}{n^2}\frac{dn}{d\omega}\right)v_g &= \frac{c}{n(k)} \end{aligned}$$

and therefore, with $\frac{kc}{n} = \omega$

$$v_g = \frac{c}{n + \omega\frac{dn}{d\omega}}$$

The wave and group velocities now differ, and depending on the sign of $\frac{dn}{d\omega}$, the group velocity may even exceed the speed of light. However, as Jackson points out, in the cases where this happens group velocity is not a useful concept.

2.3 Group velocity of a matter wave

As a final example, consider solutions to the Klein-Gordon equation, which is the relativistic form of the Schrödinger equation,

$$\Box\psi = \frac{m^2c^2}{\hbar^2}\psi$$

This has plane wave solutions of the form

$$\psi = Ae^{i(kx-\omega t)}$$

where k, ω are related to energy and momentum by

$$\begin{aligned} E &= \hbar\omega \\ p &= \hbar k \end{aligned}$$

Using the relativistic energy relation,

$$E = \sqrt{p^2c^2 + m^2c^4}$$

and substituting, we find $\omega(k)$,

$$\begin{aligned} \hbar\omega &= \sqrt{\hbar^2k^2c^2 + m^2c^4} \\ \omega &= \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}} \end{aligned}$$

The phase velocity is

$$\begin{aligned} v_{phase} &= \frac{\omega}{k} \\ &= \frac{1}{k}\sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}} \\ &= \sqrt{c^2 + \frac{m^2c^4}{\hbar^2k^2}} \\ &= c\sqrt{1 + \frac{m^2c^2}{\hbar^2k^2}} \\ &> c \end{aligned}$$

hence always greater than the speed of light, while

$$\begin{aligned} \frac{v_{group}}{c} &= \frac{1}{c}\frac{\partial\omega}{\partial k} \\ &= \frac{kc}{\sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}} \end{aligned}$$

which is always less than the speed of light.

Writing the group velocity as

$$\frac{v_{group}}{c} = \frac{kc}{\omega}$$

and identifying $v_{ph} = \frac{\omega}{k}$, we see that the product of the two velocities satisfies,

$$v_g v_{ph} = c^2$$

Notice that if we use the de Broglie relation, $p = \hbar k = \gamma m v$, the group velocity is exactly v,

$$\begin{aligned} v_{group} &= \frac{1}{\sqrt{1 + \gamma^2\frac{v^2}{c^2}}}\gamma v \\ &= \frac{1}{\sqrt{\gamma^2}}\gamma v \\ &= v \end{aligned}$$