Waves in plasma with a magnetic field

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Earth's ionosphere is a plasma, moving in Earth's magnetic field. We next consider a model for the propagation of electromagnetic waves which shows some of the effects of a magnetic field on the propagation.

1 Electromagnetic waves in plasma with a magnetic field

Consider an electromagnetic wave passing through a medium with a strong, static, uniform magnetic induction $\mathbf{B}_0$ in the same direction as the wave propagation. We neglect damping (due to collisions of the particles of the medium) and we neglect the additional magnetic field produced by the movement of the charges. Then Newton's second law for the motion of an electron with charge $-e$ becomes

$$m\ddot{\mathbf{x}} = -e\dot{\mathbf{x}} \times \mathbf{B}_0 - e\mathbf{E}e^{-i\omega t}$$

1.1 Circular polarization

Think of the waves as a superposition of the two possible circular polarizations, $\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2$,

$$\mathbf{E} = (\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)\, E_0 e^{-i\omega t}$$

To see the meaning of the circular polarization, consider the real part,

$$\begin{aligned} \mathbf{E} &= \mathcal{R}e\,(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)\, E_0 e^{-i\omega t} \\ &= \boldsymbol{\varepsilon}_1 \cos\omega t \pm \boldsymbol{\varepsilon}_2 \sin\omega t \\ &= E_0\,(\boldsymbol{\varepsilon}_1 \cos\omega t \pm \boldsymbol{\varepsilon}_2 \sin\omega t) \end{aligned}$$

The two resulting polarization vectors,

$$\begin{aligned} \boldsymbol{\varepsilon}^{+}_{\mathcal{R}e}(t) &= \boldsymbol{\varepsilon}_1 \cos\omega t + \boldsymbol{\varepsilon}_2 \sin\omega t \\ \boldsymbol{\varepsilon}^{-}_{\mathcal{R}e}(t) &= \boldsymbol{\varepsilon}_1 \cos\omega t - \boldsymbol{\varepsilon}_2 \sin\omega t \end{aligned}$$

rotate in opposite directions in the plane of the fields. The imaginary part includes the two polarizations orthogonal to these, rotating in the corresponding directions.

With the positive polarization, $\boldsymbol{\varepsilon}^{+}_{\mathcal{R}e}(t)$, the rotation is counterclockwise in the $\left(\boldsymbol{\varepsilon}_1 = \hat{\mathbf{i}}, \boldsymbol{\varepsilon}_2 = \hat{\mathbf{j}}\right)$ plane. This is called *left circularly polarized*, or *positive helicity* (but be warned: the opposite convention also is sometimes used). The second polarization rotates the opposite direction and is called *right circularly polarized*, or *negative helicity*. Positive or negative helicity correspond to whether the angular momentum vector lies parallel to, or anti-parallel to, the direction of propagation.

In the following sections, the top sign is for positive helicity and the lower sign is for negative helicity.

It is also possible to have waves with the polarizations mixed unevenly,

$$\mathbf{E} = (E_1\boldsymbol{\varepsilon}_1 \pm iE_2\boldsymbol{\varepsilon}_2)\, e^{-i\omega t}$$

These are called elliptically polarized.

1.2 Solving for the displacement of the electron

Since the magnetic field lies in the direction of propagation, it is orthogonal to both polarization vectors. Therefore, the only forces act in the plane orthogonal to the direction of propagation. Any component of velocity of the electron in the direction of the wave remains constant, so we examine only the orthogonal directions. Setting

$$\mathbf{x} = (x_1 \boldsymbol{\varepsilon}_1 \pm i x_2 \boldsymbol{\varepsilon}_2)\, e^{-i\omega t}$$

with real part,

$$\mathcal{R}e\,(\mathbf{x}) = (x_1 \cos\omega t)\, \boldsymbol{\varepsilon}_1 \pm (x_2 \sin\omega t)\, \boldsymbol{\varepsilon}_2$$

so that the position vector rotates in an ellipse. Then the equation of motion becomes

$$\begin{aligned} m\ddot{\mathbf{x}} &= -e\dot{\mathbf{x}} \times \mathbf{B}_0 - e\mathbf{E}e^{-i\omega t} \\ -\omega^2 m\,(x_1 \boldsymbol{\varepsilon}_1 \pm i x_2 \boldsymbol{\varepsilon}_2)\, e^{-i\omega t} &= +i\omega e\,(x_1 \boldsymbol{\varepsilon}_1 \pm i x_2 \boldsymbol{\varepsilon}_2)\, e^{-i\omega t} \times \mathbf{B}_0 - e\,(\boldsymbol{\varepsilon}_1 \pm i \boldsymbol{\varepsilon}_2)\, E e^{-i\omega t} \end{aligned}$$

Expanding the cross product term,

$$\begin{aligned} i\omega e\,(x_1 \boldsymbol{\varepsilon}_1 \pm i x_2 \boldsymbol{\varepsilon}_2)\, e^{-i\omega t} \times \mathbf{B}_0 &= i\omega e\,(x_1 \boldsymbol{\varepsilon}_1 \times \mathbf{B}_0 \pm i x_2 \boldsymbol{\varepsilon}_2 \times \mathbf{B}_0)\, e^{-i\omega t} \\ &= i\omega e\,(-x_1 \boldsymbol{\varepsilon}_2 \pm i x_2 \boldsymbol{\varepsilon}_1)\, B_0 e^{-i\omega t} \end{aligned}$$

we write the separate components

$$\begin{aligned} -\omega^2 m\,(x_1 \boldsymbol{\varepsilon}_1) &= +i\omega e\,(\pm i x_2 \boldsymbol{\varepsilon}_1)\, B_0 - e\boldsymbol{\varepsilon}_1 E \\ -\omega^2 m\,(\pm i x_2 \boldsymbol{\varepsilon}_2) &= +i\omega e\,(-x_1 \boldsymbol{\varepsilon}_2)\, B_0 - e\,(\pm i \boldsymbol{\varepsilon}_2)\, E \end{aligned}$$

The first gives

$$\begin{aligned} -\omega^2 m x_1 &= \mp \omega e x_2 \boldsymbol{\varepsilon}_1 B_0 - eE \\ x_1 &= \pm \frac{eB_0}{m\omega} x_2 + \frac{eE}{m\omega^2} \end{aligned}$$

and the second,

$$\begin{aligned} \pm i\omega^2 m x_2 &= +i\omega e x_1 B_0 \pm ieE \\ x_2 &= \pm \frac{eB_0}{m\omega} x_1 + \frac{eE}{m\omega^2} \end{aligned}$$

Now substitute the first into the second to find

$$\begin{aligned} x_2 &= \pm \frac{eB_0}{\omega m} \left(\pm \frac{eB_0}{m\omega} x_2 + \frac{eE}{m\omega^2} \right) + \frac{eE}{m\omega^2} \\ x_2 &= \left(\frac{eB_0}{\omega m} \right)^2 x_2 \pm \frac{e^2 B_0 E}{m^2 \omega^3} + \frac{eE}{m\omega^2} \\ \left(1 - \left(\frac{eB_0}{\omega m} \right)^2 \right) x_2 &= \frac{eE}{m\omega^2} \left(1 \pm \frac{eB_0}{m\omega} \right) \\ x_2 &= \frac{eE}{m\omega^2} \frac{1 \pm \frac{eB_0}{m\omega}}{1 - \left(\frac{eB_0}{\omega m} \right)^2} \end{aligned}$$

Define the precession frequency,

$$\omega_B \equiv \frac{eB_0}{m}$$

Then, factoring the denominator on the right, we have

$$\begin{aligned} x_2 &= \frac{eE}{m\omega^2}\frac{1\pm\frac{\omega_B}{\omega}}{\left(1-\frac{\omega_B}{\omega}\right)\left(1+\frac{\omega_B}{\omega}\right)} \\ &= \frac{eE}{m\omega^2}\frac{1}{1\mp\frac{\omega_B}{\omega}} \\ &= \frac{eE}{m\omega}\frac{1}{\omega\mp\omega_B} \end{aligned}$$

for x_2. The result for x_1 is therefore,

$$\begin{aligned} x_1 &= \pm\frac{eB_0}{m\omega}x_2+\frac{eE}{m\omega^2} \\ &= \frac{eE}{m\omega^2}\left(1\pm\frac{\omega_B}{\omega\mp\omega_B}\right) \\ &= \frac{eE}{m\omega^2}\left(\frac{\omega\mp\omega_B\pm\omega_B}{\omega\mp\omega_B}\right) \\ &= \frac{eE}{m\omega}\left(\frac{1}{\omega\mp\omega_B}\right) \end{aligned}$$

so reconstructing the full position vector,

$$\begin{aligned} \mathbf{x} &= \left(x_1\boldsymbol{\varepsilon}_1\pm ix_2\boldsymbol{\varepsilon}_2\right)e^{-i\omega t} \\ &= \left(\frac{eE}{m\omega}\left(\frac{1}{\omega\mp\omega_B}\right)\boldsymbol{\varepsilon}_1\pm i\frac{eE}{m\omega}\frac{1}{\omega\mp\omega_B}\boldsymbol{\varepsilon}_2\right)e^{-i\omega t} \\ &= \frac{e}{m\omega}\left(\frac{1}{\omega\mp\omega_B}\right)\left(\boldsymbol{\varepsilon}_1\pm i\boldsymbol{\varepsilon}_2\right)Ee^{-i\omega t} \end{aligned}$$

and finally,

$$\mathbf{x} = \frac{e}{m\omega}\left(\frac{1}{\omega\mp\omega_B}\right)\mathbf{E}$$

Physically, what happens is that each circular polarization drives the electron position in a corresponding circle, with the amplitude of the circle diverging if the wave is at the precession frequency of the electron. Notice the very different behavior of the two polarizations.

2 Dielectric constant

The dielectric constant is found in the same way as for our previous model. We repeat the argument with this new solution.

The electric dipole moment is

$$\begin{aligned} \mathbf{p} &= -e\mathbf{x} \\ &= -\frac{e^2}{m\omega}\left(\frac{1}{\omega\mp\omega_B}\right)\mathbf{E} \end{aligned}$$

Let there be N molecules per unit volume with Z electrons per molecule, with a fraction f_i of the electrons having binding frequency ω_{i0} and damping γ_i, accounting for the different binding energies of the different electrons. The total of all the f_i should be the total number of electrons, $\sum_i f_i = Z$. The dipole moment for each molecule is then

$$\mathbf{p}_{mol} = -\frac{e^2}{m\omega}\sum_i \frac{f_i}{\omega\mp\omega_B}\mathbf{E}$$

Then, since the total dipole moment per unit volume is $\mathbf{P} = N\mathbf{p}_{mol} = \varepsilon_0 \chi_e \mathbf{E}$, the dielectric constant is

$$\begin{aligned} \epsilon &= 1 + \chi_e \\ &= 1 - \frac{Ne^2}{m\omega\epsilon_0} \sum_i \frac{f_i}{\omega \mp \omega_B} \end{aligned}$$

At high frequency, this becomes

$$\begin{aligned} \epsilon &= 1 - \frac{Ne^2}{m\omega\epsilon_0} \frac{\sum_i f_i}{\omega \mp \omega_B} \\ &= 1 - \frac{NZe^2}{m\omega\epsilon_0} \frac{1}{\omega \mp \omega_B} \\ &= 1 - \frac{\omega_P^2}{\omega(\omega \mp \omega_B)} \end{aligned}$$

where we again find the plasma frequency, $\omega_p^2 = \frac{NZe^2}{m\epsilon_0}$.

3 Wave vector

Next, we find the wave vector corresponding to this dielectric constant. With $\mu = \mu_0$, we have

$$\begin{aligned} kc &= \sqrt{\mu\varepsilon}\omega \\ &= \sqrt{\epsilon}\omega \\ &= \omega\sqrt{1 - \frac{\omega_P^2}{\omega(\omega \mp \omega_B)}} \end{aligned}$$

which goes imaginary whenever $\frac{\omega_P^2}{\omega(\omega \mp \omega_B)} > 1$. The imaginary wave vector gives exponential damping of the wave, a result of strong atomic absorption of the energy.

For the upper, positive helicity sign, first notice that when $\omega < \omega_B$ we have $1 - \frac{\omega_P^2}{\omega(\omega - \omega_B)} = 1 + \frac{\omega_P^2}{\omega(\omega_B - \omega)} > 0$ and there is no damping. When $\omega > \omega_B$, the propagation is damped whenever

$$\omega^2 - \omega_B\omega - \omega_P^2 < 0$$

This occurs when the frequency is below $\omega = \frac{1}{2}\left(\omega_B + \sqrt{\omega_B^2 + 4\omega_P^2}\right)$, i.e.,

$$\omega < \frac{\omega_B}{2}\left(1 + \sqrt{1 + \frac{4\omega_P^2}{\omega_B^2}}\right)$$

Damping of positive helicity waves therefore occurs when the frequency satisfies

$$\omega_B < \omega < \frac{\omega_B}{2}\left(1 + \sqrt{1 + \frac{4\omega_P^2}{\omega_B^2}}\right)$$

For negative helicity waves, the condition becomes

$$\omega^2 + \omega\omega_B - \omega_P^2 < 0$$

which occurs for frequencies below

$$\omega = \frac{1}{2}\left(-\omega_B \pm \sqrt{\omega_B^2 + 4\omega_P^2}\right)$$

that is,

$$\omega < \frac{\omega_B}{2}\left(\sqrt{1 + \frac{4\omega_P^2}{\omega_B^2}} - 1\right)$$

4 The ionosphere

An electromagnetic pulse sent up into the ionosphere will reflect if one of these condtions is met for the appropriate polarization. Since the plasma frequency,

$$\omega_p = \sqrt{\frac{NZe^2}{m\epsilon_0}}$$

varies as the square root of the number density of electrons, we can measure the electron density by timing the round trip travel time of the pulse as a function of frequency.

Consider an experiment in which we measure the longest round trip travel time as a function of the the frequency of some negative helicity pulses, giving $T(\omega)$. This value translates directly into altitude, $h = h(T(\omega))$. Inverting, we have $\omega(h)$. Now, solving for the plasma frequency, at the extremal value for total reflection,

$$\begin{aligned} \omega^2 + \omega\omega_B - \omega_P^2 &= 0 \\ \omega_P &= \sqrt{\omega^2 + \omega\omega_B} \end{aligned}$$

we have

$$\sqrt{\frac{NZe^2}{m\epsilon_0}} = \sqrt{\omega^2 + \omega\omega_B}$$

and therefore,

$$N(h) = \frac{m\epsilon_0}{Ze^2}\left((\omega(h))^2 + \omega(h)\,\omega_B\right)$$

The resulting curves show a general increase due to increasing ionization with altitude, then an ultimate decrease as the atmospheric density drops off. There is a great deal of structure to the curves, and they vary from day to night and depend on solar activity.