

Frequency dependence of the permittivity

February 7, 2016

In materials, the dielectric “constant” and permeability are actually frequency dependent. This does not affect our results for single frequency modes, but when we have a superposition of frequencies it leads to dispersion. We begin with a simple model for this behavior. The variation of the permeability is often quite weak, and we may take $\mu = \mu_0$.

1 Frequency dependence of the permittivity

1.1 Permittivity produced by a static field

The electrostatic treatment of the dielectric constant begins with the electric dipole moment produced by an electron in a static electric field \mathbf{E} . The electron experiences a linear restoring force, $\mathbf{F} = -m\omega_0^2\mathbf{x}$,

$$e\mathbf{E} = m\omega_0^2\mathbf{x}$$

where ω_0 characterizes the strength of the atom’s restoring potential. The resulting displacement of charge, $\mathbf{x} = \frac{e\mathbf{E}}{m\omega_0^2}$, produces a molecular polarization

$$\begin{aligned}\mathbf{p}_{mol} &= e\mathbf{x} \\ &= \frac{e\mathbf{E}}{m\omega_0^2}\end{aligned}$$

Then, if there are N molecules per unit volume with Z electrons per molecule, the dipole moment per unit volume is

$$\mathbf{P} = NZ\mathbf{p}_{mol} \equiv \epsilon_0\chi_e\mathbf{E}$$

so that

$$\epsilon_0\chi_e = \frac{NZe^2}{m\omega_0^2}$$

Next, using

$$\begin{aligned}\mathbf{D} &= \epsilon_0\mathbf{E} + \mathbf{P} \\ \epsilon\mathbf{E} &= \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E}\end{aligned}$$

the relative dielectric constant is

$$\begin{aligned}\epsilon = \frac{\epsilon}{\epsilon_0} &= 1 + \chi_e \\ &= 1 + \frac{NZe^2}{m\omega_0^2\epsilon_0}\end{aligned}$$

This result changes when there is time dependence to the electric field, with the dielectric constant showing frequency dependence.

1.2 Permittivity in the presence of an oscillating electric field

Suppose the material is sufficiently diffuse that the applied electric field is about equal to the electric field at each atom, and that the response of the atomic electrons may be modeled as harmonic. Again let \mathbf{x} represent the displacement of the charge from equilibrium, and include damping, so that now we have

$$m [\ddot{\mathbf{x}} + \gamma \dot{\mathbf{x}} + \omega_0^2 \mathbf{x}] = -e\mathbf{E}(\mathbf{x}, t)$$

In addition to the linear response of the atom, we assume negligible magnetic effects and low-amplitude oscillations. The model is still enough to give important general features.

Let the electric field vary harmonically,

$$\mathbf{E} = \mathbf{E}(\mathbf{x}) e^{-i\omega t}$$

then the position of the electron will have the same time dependence, $\mathbf{x}(t) = \mathbf{x}e^{-i\omega t}$, so

$$m [-\omega^2 - i\omega\gamma + \omega_0^2] \mathbf{x} = -e\mathbf{E}(\mathbf{x})$$

and the electric dipole moment is

$$\begin{aligned} \mathbf{p} &= -e\mathbf{x} \\ &= \frac{e^2 \mathbf{E}}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \end{aligned}$$

Let there be N molecules per unit volume with Z electrons per molecule, with a fraction f_i of the electrons having binding frequency ω_{i0} and damping γ_i . This is reasonable since the different electrons in each molecule are bound differently to the nucleus. The total of all the f_i should be the total number of electrons, $\sum_i f_i = Z$. The dipole moment for each molecule is then

$$\mathbf{p}_{mol} = \sum_i \frac{f_i e^2}{m(\omega_{i0}^2 - \omega^2 - i\omega\gamma_i)} \mathbf{E}$$

Then, since the total dipole moment per unit volume is $\mathbf{P} = N\mathbf{p}_{mol} = \epsilon_0 \chi_e \mathbf{E}$, we have

$$\sum_i \frac{N f_i e^2}{m(\omega_{i0}^2 - \omega^2 - i\omega\gamma_i)} \mathbf{E} = \epsilon_0 \chi_e \mathbf{E}$$

Now, using $\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E}$, the relative dielectric constant is

$$\begin{aligned} \epsilon = \frac{\epsilon}{\epsilon_0} &= 1 + \chi_e \\ &= 1 + \frac{N}{\epsilon_0} \sum \frac{f_i e^2}{m(\omega_{i0}^2 - \omega^2 - i\omega\gamma_i)} \\ &= 1 + \frac{N e^2}{\epsilon_0 m} \sum \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \end{aligned}$$

This expression is accurate if f_i, ω_{i0} and γ_i are found quantum mechanically.

1.3 Anomalous dispersion and resonant absorption

The frequency dependence of the dielectric constant has certain regular properties. Separating real and imaginary parts,

$$\begin{aligned} \epsilon &= 1 + \frac{N e^2}{m \epsilon_0} \sum \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \\ &= 1 + \frac{N e^2}{m \epsilon_0} \sum \frac{f_i (\omega_{i0}^2 - \omega^2 + i\omega\gamma_i)}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2 \gamma_i^2} \end{aligned}$$

we have

$$\begin{aligned} \text{Re } \epsilon &= 1 + \frac{Ne^2}{m\epsilon_0} \sum \frac{f_i (\omega_{i0}^2 - \omega^2)}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2 \gamma_i^2} \\ \text{Im } \epsilon &= \frac{Ne^2 \omega}{m\epsilon_0} \sum \frac{f_i \gamma_i}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2 \gamma_i^2} \end{aligned}$$

and we note that the damping constant γ_i is usually small. At low frequencies,

$$\omega < \omega_{i0}$$

for all i , so each term in the real part of ϵ is positive and $\epsilon > 1$. As the frequency increases, more and more of the terms become negative, until at high frequency,

$$\text{Re } \epsilon = 1 - \frac{Ne^2}{m\epsilon_0} \sum \frac{f_i |\omega^2 - \omega_{i0}^2|}{(\omega_{i0}^2 - \omega^2)^2 + \omega^2 \gamma_i^2}$$

and $\epsilon < 1$.

The real and imaginary parts of ϵ have peaks whenever ω is near one of the resonant frequencies, ω_{i0} , of the molecule. At these frequencies, the corresponding term of the denominator becomes

$$(\omega_{i0}^2 - \omega^2)^2 + \omega^2 \gamma_i^2 \rightarrow \omega^2 \gamma_i^2$$

which is very small (but larger the higher the driving frequency). Since the resonant real part is changing sign, the peak is double, spiking positive for $\omega < \omega_{i0}$ and then spiking negative once $\omega > \omega_{i0}$. At the same time, the imaginary part of ϵ also has a peak, causing the material to absorb energy from the field. Clearly, the damping γ_i plays an important role, governing the magnitude of the resonances. These effects are easily seen from the wave vector \mathbf{k} .

Suppose we have a plane wave travelling in the z direction, $\mathbf{E} = \mathcal{E} e^{i(kz - \omega t)}$, which passes near a resonant atom. Then setting $\mu \approx 1$,

$$\begin{aligned} k &= \omega \sqrt{\mu \epsilon} \\ &= \omega \sqrt{\text{Re } \epsilon + i \text{Im } \epsilon} \end{aligned}$$

We need the real and imaginary parts of k , so set

$$k \equiv \beta + \frac{1}{2} i \alpha$$

The factor of $\frac{1}{2}$ is helpful below. We have

$$\begin{aligned} \beta + \frac{1}{2} i \alpha &= \omega \sqrt{\text{Re } \epsilon + i \text{Im } \epsilon} \\ \beta^2 + i \alpha \beta - \frac{1}{4} \alpha^2 &= \omega^2 (\text{Re } \epsilon + i \text{Im } \epsilon) \end{aligned}$$

so that

$$\begin{aligned} \beta^2 - \frac{1}{4} \alpha^2 &= \omega^2 \text{Re } \epsilon \\ \alpha \beta &= \omega^2 \text{Im } \epsilon \end{aligned}$$

and the electric field is given by

$$\begin{aligned} \mathbf{E} &= \mathcal{E} e^{i(kz - \omega t)} \\ &= \mathcal{E} e^{-\frac{1}{2} \alpha z} e^{i(\beta z - \omega t)} \end{aligned}$$

The intensity of the wave, which varies as \mathbf{E}^2 then falls off as

$$e^{-\alpha z}$$

and α is called the attenuation constant. We see that the attenuation is larger near each resonant frequency, ω_{i0} , where we expect the radiation to be exciting the electrons of the molecule. Relating this back to the resonance behavior, with $\alpha \ll \beta$,

$$\begin{aligned}\beta^2 &\approx \beta^2 - \frac{1}{4}\alpha^2 = \omega^2 Re \epsilon \\ \alpha &= \frac{\omega^2}{\beta} Im \epsilon \\ &\approx \frac{\omega}{\sqrt{Re \epsilon}} Im \epsilon\end{aligned}$$

Suppose, for simplicity, we have only a single resonance. Then with $\omega \approx \omega_0$

$$\begin{aligned}Re \epsilon &= 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \\ &\approx 1 \\ Im \epsilon &= \frac{Ne^2\omega}{m\epsilon_0} \frac{\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \\ &\approx \frac{Ne^2}{m\epsilon_0\omega\gamma}\end{aligned}$$

so the attenuation constant is approximately

$$\alpha = \frac{Ne^2}{m\epsilon_0\gamma}$$

2 Low frequencies: conductivity

In a conductor there are free electrons. Since these have no restoring force, they may be thought of as having a resonance frequency of zero. This makes the response of conductors at low frequency very different from that of insulators.

For insulators, the lowest resonant frequency is $\omega_{10} > 0$ and the equations above give a good approximation to the actual response.

For conductors, there is a very strong response at zero frequency. By including a zero frequency mode in the dielectric constant, we can derive Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, as an effect of the resulting imaginary part of the dielectric constant.

Separate out the fraction of free electrons, f_0 , in our expression for the relative dielectric constant,

$$\begin{aligned}\epsilon &= 1 + \frac{Ne^2}{m\epsilon_0} \sum_{i=0}^n \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \\ &= 1 + \frac{Ne^2}{m\epsilon_0} \frac{f_0}{-\omega^2 - i\omega\gamma_0} + \frac{Ne^2}{m\epsilon_0} \sum_{i=1}^n \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \\ &= \epsilon_b + i \frac{Ne^2 f_0}{m\epsilon_0\omega(\gamma_0 - i\omega)}\end{aligned}$$

where we define the background dielectric constant, $\epsilon_b \equiv 1 + \frac{Ne^2}{m\epsilon_0} \sum_{i=1}^n \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i}$. To see what is happening in terms of conduction, consider the Maxwell equation involving current,

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$$

Applying this for harmonic fields and a real dielectric constant, ϵ_b , and no current gives

$$\begin{aligned}\frac{1}{\epsilon_0} \nabla \times \mathbf{H} - \frac{\partial (\epsilon_b \mathbf{E})}{\partial t} &= 0 \\ \frac{i}{\mu \epsilon_0} \mathbf{k} \times \mathbf{H} + i\omega \epsilon_b \mathbf{E} &= 0\end{aligned}$$

Now include the zero mode of the dielectric constant,

$$\begin{aligned}\frac{1}{\epsilon_0} \nabla \times \mathbf{H} - \frac{\partial (\epsilon \mathbf{E})}{\partial t} &= 0 \\ \frac{i}{\mu \epsilon_0} \mathbf{k} \times \mathbf{H} + i\omega \left(\epsilon_b + i \frac{Ne^2 f_0}{m \epsilon_0 \omega (\gamma_0 - i\omega)} \right) \mathbf{E} &= 0\end{aligned}$$

or, defining the conductivity as

$$\sigma \equiv \frac{Ne^2 f_0}{m \epsilon_0 (\gamma_0 - i\omega)}$$

we have

$$\frac{i}{\mu \epsilon_0} \mathbf{k} \times \mathbf{H} + i\omega \epsilon_b \mathbf{E} = \sigma \mathbf{E}$$

Comparing this to the full harmonic form of Ampère's law,

$$\frac{i}{\mu \epsilon_0} \mathbf{k} \times \mathbf{H} + i\omega \epsilon_b \mathbf{E} = \mathbf{J}$$

we have derived the form of Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E}$$

where σ is no longer a phenomenological constant, but arises from zero modes of the material.

Our expression for the conductivity,

$$\sigma = \frac{Ne^2 f_0}{m (\gamma_0 - i\omega)}$$

was first produced by Drude in 1900. It requires substantial correction because the free electrons actually form a Fermi gas.

3 High frequencies: The plasma frequency

Now we expand our expression for the dielectric constant for $\omega \gg \omega_{i0}$ for all i . The denominator is approximately $\omega_{i0}^2 - \omega^2 - i\omega\gamma_i \approx -\omega^2$ for all modes and we have

$$\begin{aligned}\epsilon &= 1 + \frac{Ne^2}{m \epsilon_0} \sum \frac{f_i}{\omega_{i0}^2 - \omega^2 - i\omega\gamma_i} \\ &\approx 1 - \frac{Ne^2}{m \omega^2 \epsilon_0} \sum f_i\end{aligned}$$

Replacing the sum over fractions by the atomic number, $\sum f_i = Z$,

$$\begin{aligned}\epsilon &= 1 - \frac{NZe^2}{m \omega^2 \epsilon_0} \\ &= 1 - \frac{\omega_p^2}{\omega^2}\end{aligned}$$

where

$$\omega_p \equiv \sqrt{\frac{NZe^2}{m\epsilon_0}}$$

is called the plasma frequency. The plasma frequency depends only on NZ , the total number of electrons in the system. Combining with the wave number k , we have

$$\begin{aligned}k &= \sqrt{\mu\epsilon} \frac{\omega}{c} \\k^2 &= \frac{1}{c^2} \omega^2 \epsilon \\ &= \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2}\right)\end{aligned}$$

so that

$$\omega^2 = k^2 c^2 + \omega_p^2$$

This provides a dispersion relation, $\omega(k)$, for plasmas. We will explore more about dispersion relations soon.

3.1 Example: Water

Jackson has collected all available data (to the date of first publication, anyway) on the index of refraction and attenuation coefficient of water as functions of frequency. The graphs on page 315 are fascinating! Take some time to digest what is going on. The window in the visible range is particularly interesting.