Reflection and refraction of plane waves at an interface

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1 Boundary conditions

Consider space filled with two uniform linear materials, one with permitivity and permeability, ϵ, μ and the other with ϵ', μ'. Let the first occupy all space below the xy-plane and the other all space above the xy-plane. The usual electromagnetic boundary conditions therefore apply at the interface at $z = 0$.

1.1 Geometry of the waves

Let the incident wave move with wave vector, $\mathbf{k}$, frequency ω, and fields $\mathbf{E}, \mathbf{B}$. Denote the same quantities for the refracted wave by $\mathbf{k}', \omega', \mathbf{E}', \mathbf{B}'$, and those for the reflected wave by $\mathbf{k}'', \omega'', \mathbf{E}'', \mathbf{B}''$.

In order to avoid discontinuity, and buildup of energy at the boundary, the space and time dependence of the incoming, reflected, and refracted waves must match,

$$
\begin{aligned}
e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\Big|_{z=0} &= e^{i\left(\mathbf{k}'\cdot\mathbf{x}-\omega' t\right)}\Big|_{z=0} = e^{i\left(\mathbf{k}''\cdot\mathbf{x}-\omega'' t\right)}\Big|_{z=0} \\
\mathbf{k}\cdot\mathbf{x}-\omega t\big|_{z=0} &= \mathbf{k}'\cdot\mathbf{x}-\omega' t\big|_{z=0} = \mathbf{k}''\cdot\mathbf{x}-\omega'' t\big|_{z=0}
\end{aligned}
$$

These relations have to hold at all times and at all x and y. Therefore,

$$\omega = \omega' = \omega''$$

and

$$k_x x + k_y y = k_x^{'} x + k_y^{'} y = k_x^{''} x + k_y^{''} y$$

We choose the incoming wave to lie in the xz-plane, so that $k_y = 0$. Let the incoming wave make an angle, i, with the normal to the interface – the angle of incidence. Let the refracted wave make an acute angle r and the reflected wave an acute angle r' with the normal. Varying x and y independently, it immediately follows that $k_y^{'} = 0$ and $k_y^{''} = 0$, so the three waves are coplanar, and the x-components equal, $k_x = k_x' = k_x''$. Writing the x components in terms of the full magnitude of the wave vector, we have

$$k \sin i = k' \sin r = k'' \sin r'$$

Since the incident and reflected waves are in the same medium, $k = k''$, and therefore the angle of reflection equals the angle of incidence,

$$r' = i$$

while, using the relationship for index of refraction, $n = \frac{c}{v} = \sqrt{\mu\epsilon} = \frac{c}{\omega}k$, we may write the ratio of wave vectors as the ratio of the indices of refraction,

$$\frac{k'}{k} = \frac{n'}{n}$$

and therefore, Snell's law for the angle of refraction,

$$n \sin i = n' \sin r$$

1.2 Relations between the fields

Now we turn to the boundary conditions. With $\hat{\mathbf{n}}$ the unit normal to the interface (that is, a unit vector in the positive z direction), we have:

1. Continuity of the normal component of $\mathbf{D}$,

$$\epsilon \left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}'' \right) \cdot \hat{\mathbf{n}} = \epsilon' \boldsymbol{\mathcal{E}}' \cdot \hat{\mathbf{n}}$$

2. Continuity of the tangential component of $\mathbf{E}$,

$$\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}'' \right) \times \hat{\mathbf{n}} = \boldsymbol{\mathcal{E}}' \times \hat{\mathbf{n}}$$

3. Continuity of the normal component of $\mathbf{B}$. Using $\boldsymbol{\mathcal{B}} = \frac{1}{k}\sqrt{\mu\epsilon}\mathbf{k} \times \boldsymbol{\mathcal{E}} = \frac{1}{\omega}\mathbf{k} \times \boldsymbol{\mathcal{E}}$ and cancelling the overall ω,

$$\begin{aligned} \left(\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}'' \right) \cdot \hat{\mathbf{n}} &= \boldsymbol{\mathcal{B}}' \cdot \hat{\mathbf{n}} \\ \left(\mathbf{k} \times \boldsymbol{\mathcal{E}} + \mathbf{k}'' \times \boldsymbol{\mathcal{E}}'' \right) \cdot \hat{\mathbf{n}} &= \mathbf{k}' \times \boldsymbol{\mathcal{E}}' \cdot \hat{\mathbf{n}} \end{aligned}$$

4. Continuity of the tangential component of $\mathbf{H}$,

$$\frac{1}{\mu} \left(\mathbf{k} \times \boldsymbol{\mathcal{E}} + \mathbf{k}'' \times \boldsymbol{\mathcal{E}}'' \right) \times \hat{\mathbf{n}} = \frac{1}{\mu'} \left(\mathbf{k}' \times \boldsymbol{\mathcal{E}}' \right) \times \hat{\mathbf{n}}$$

Instead of applying these equations to a general incident polarization, we may treat the two independent linear polarization directions independently, with the general result being a linear combination of the two. We therefore need consider only two particular cases,

1. The electric field perpendicular to the plane incidence (i.e., the $+y$-direction)
2. The electric field parallel to the plane of incidence.

We continue to take the plane of incidence to be the xz-plane, the plane of the interface to be the xy plane at $z = 0$, and $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ the forward normal to the interface.

2 Polarization perpendicular to the plane of incidence

Let the electric field vector, $\boldsymbol{\mathcal{E}}$, point in the positive y direction. Setting

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \hat{\mathbf{j}}\mathcal{E} \\ \boldsymbol{\mathcal{E}}' &= \hat{\mathbf{j}}\mathcal{E}' \\ \boldsymbol{\mathcal{E}}'' &= \hat{\mathbf{j}}\mathcal{E}'' \end{aligned}$$

and the wave vectors as

$$\begin{aligned} \mathbf{k} &= k \sin i \hat{\mathbf{i}} + k \cos i \hat{\mathbf{k}} \\ \mathbf{k}' &= k' \sin r \hat{\mathbf{i}} + k' \cos r \hat{\mathbf{k}} \\ \mathbf{k}'' &= k \sin i \hat{\mathbf{i}} - k \cos i \hat{\mathbf{k}} \end{aligned}$$

the first boundary condition is satisfied identically. Substituting the expansions into the remaining boundary conditions,

$$
\begin{aligned}
\left(\hat{\mathbf{j}}\mathcal{E} + \hat{\mathbf{j}}\mathcal{E}''\right) \times \hat{\mathbf{k}} &= \hat{\mathbf{j}}\mathcal{E}' \times \hat{\mathbf{k}} \\
\left(\left(k\sin i\hat{\mathbf{i}} + k\cos i\hat{\mathbf{k}}\right) \times \hat{\mathbf{j}}\mathcal{E} + \left(k\sin i\hat{\mathbf{i}} - k\cos i\hat{\mathbf{k}}\right) \times \hat{\mathbf{j}}\mathcal{E}''\right) \cdot \hat{\mathbf{k}} &= \left(k'\sin r\hat{\mathbf{i}} + k'\cos r\hat{\mathbf{k}}\right) \times \hat{\mathbf{j}}\mathcal{E}' \cdot \hat{\mathbf{k}} \\
\frac{1}{\mu}\left(\left(k\sin i\hat{\mathbf{i}} + k\cos i\hat{\mathbf{k}}\right) \times \hat{\mathbf{j}}\mathcal{E} + \left(k\sin i\hat{\mathbf{i}} - k\cos i\hat{\mathbf{k}}\right) \times \hat{\mathbf{j}}\mathcal{E}''\right) \times \hat{\mathbf{k}} &= \frac{1}{\mu'}\left(\left(k'\sin r\hat{\mathbf{i}} + k'\cos r\hat{\mathbf{k}}\right) \times \hat{\mathbf{j}}\mathcal{E}'\right) \times \hat{\mathbf{k}}
\end{aligned}
$$

and distributing,

$$
\begin{aligned}
\left(\mathcal{E} + \mathcal{E}''\right)\hat{\mathbf{i}} &= \mathcal{E}'\hat{\mathbf{i}} \\
\left(k\mathcal{E}\sin i\hat{\mathbf{i}} \times \hat{\mathbf{j}} + k\mathcal{E}\cos i\hat{\mathbf{k}} \times \hat{\mathbf{j}} + k\mathcal{E}''\sin i\hat{\mathbf{i}} \times \hat{\mathbf{j}} - k\mathcal{E}''\cos i\hat{\mathbf{k}} \times \hat{\mathbf{j}}\right) \cdot \hat{\mathbf{k}} &= \left(k'\mathcal{E}'\sin r\hat{\mathbf{i}} \times \hat{\mathbf{j}} + k'\mathcal{E}'\cos r\hat{\mathbf{k}} \times \hat{\mathbf{j}}\right) \cdot \hat{\mathbf{k}} \\
\left(\frac{1}{\mu}k\mathcal{E}\sin i\hat{\mathbf{i}} \times \hat{\mathbf{j}} + \frac{1}{\mu}k\mathcal{E}\cos i\hat{\mathbf{k}} \times \hat{\mathbf{j}} + \frac{1}{\mu}k\mathcal{E}''\sin i\hat{\mathbf{i}} \times \hat{\mathbf{j}} - \frac{1}{\mu}k\mathcal{E}''\cos i\hat{\mathbf{k}} \times \hat{\mathbf{j}}\right) \times \hat{\mathbf{k}} &= \frac{1}{\mu'}\left(k'\mathcal{E}'\sin r\hat{\mathbf{i}} \times \hat{\mathbf{j}} + k'\mathcal{E}'\cos r\hat{\mathbf{k}} \times \hat{\mathbf{j}}\right) \times \hat{\mathbf{k}}
\end{aligned}
$$

we carry out the dot and cross products,

$$
\begin{aligned}
\mathcal{E} + \mathcal{E}'' &= \mathcal{E}' \\
k\mathcal{E}\sin i + k\mathcal{E}''\sin i &= k'\mathcal{E}'\sin r \\
-\frac{1}{\mu}k\mathcal{E}\cos i + \frac{1}{\mu}k\mathcal{E}''\cos i &= -\frac{1}{\mu'}k'\mathcal{E}'\cos r
\end{aligned}
$$

Using Snell's law in the form $k\sin i = k'\sin r$, the first two of these are identical, while using

$$
\frac{k}{\mu} = \frac{\omega}{c}\sqrt{\frac{\epsilon}{\mu}}
$$

to express the result in terms of properties of the medium, then cancelling an overall factor of $\frac{\omega}{c}$, the third becomes

$$
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E} - \mathcal{E}''\right)\cos i = \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'\cos r
$$

Eliminating $\mathcal{E}'$ using the first equation, we solve for $\frac{\mathcal{E}''}{\mathcal{E}}$,

$$
\begin{aligned}
\sqrt{\frac{\epsilon}{\mu}}\mathcal{E}\cos i - \sqrt{\frac{\epsilon}{\mu}}\mathcal{E}''\cos i &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}\cos r + \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}''\cos r \\
-\sqrt{\frac{\epsilon}{\mu}}\mathcal{E}''\cos i - \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}''\cos r &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}\cos r - \sqrt{\frac{\epsilon}{\mu}}\mathcal{E}\cos i \\
\mathcal{E}''\left(\sqrt{\frac{\epsilon}{\mu}}\cos i + \sqrt{\frac{\epsilon'}{\mu'}}\cos r\right) &= \mathcal{E}\left(\sqrt{\frac{\epsilon}{\mu}}\cos i - \sqrt{\frac{\epsilon'}{\mu'}}\cos r\right)
\end{aligned}
$$

so that

$$
\frac{\mathcal{E}''}{\mathcal{E}} = \frac{\sqrt{\frac{\epsilon}{\mu}}\cos i - \sqrt{\frac{\epsilon'}{\mu'}}\cos r}{\sqrt{\frac{\epsilon}{\mu}}\cos i + \sqrt{\frac{\epsilon'}{\mu'}}\cos r}
$$

Next we want to express this in terms of the index of refraction,

$$
n = \frac{c}{v} = \sqrt{\mu\epsilon}
$$

so that we replace

$$\sqrt{\frac{\epsilon}{\mu}} = \frac{n}{\mu}$$

and similarly $\sqrt{\frac{\epsilon'}{\mu'}} = \frac{n'}{\mu'}$,

$$\begin{aligned} \frac{\mathcal{E}''}{\mathcal{E}} &= \frac{\frac{n}{\mu}\cos i - \frac{n'}{\mu'}\cos r}{\frac{n}{\mu}\cos i + \frac{n'}{\mu'}\cos r} \\ &= \frac{n\cos i - \frac{\mu}{\mu'}n'\cos r}{n\cos i + \frac{\mu}{\mu'}n'\cos r} \end{aligned}$$

This is often useful because frequently $\mu \approx \mu'$ and we know the indices of refraction. We would also like to express the result in terms of the angle of incidence, so we replace

$$\begin{aligned} \cos r &= \sqrt{1-\sin^2 r} \\ &= \sqrt{1-\frac{n^2}{n'^2}\sin^2 i} \\ &= \frac{1}{n'}\sqrt{n'^2 - n^2\sin^2 i} \end{aligned}$$

This gives

$$\frac{\mathcal{E}''}{\mathcal{E}} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

Finally, substituting this result into $\mathcal{E}' = \mathcal{E} + \mathcal{E}''$, we have

$$\begin{aligned} \frac{\mathcal{E}'}{\mathcal{E}} &= 1 + \frac{\mathcal{E}''}{\mathcal{E}} \\ &= 1 + \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\ &= \frac{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i} + n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\ &= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \end{aligned}$$

so we have the final result for parallel reflection/refraction:

$$\begin{aligned} \frac{\mathcal{E}'}{\mathcal{E}} &= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\ \frac{\mathcal{E}''}{\mathcal{E}} &= \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \end{aligned}$$

3 Polarization in the plane of incidence

Now let $\boldsymbol{\mathcal{E}}$ lie in the xz-plane. Then, setting

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= -\hat{\mathbf{i}}\mathcal{E}\cos i + \hat{\mathbf{k}}\mathcal{E}\sin i \\ \boldsymbol{\mathcal{E}}' &= -\hat{\mathbf{i}}\mathcal{E}'\cos r + \hat{\mathbf{k}}\mathcal{E}'\sin r \\ \boldsymbol{\mathcal{E}}'' &= \hat{\mathbf{i}}\mathcal{E}''\cos i + \hat{\mathbf{k}}\mathcal{E}''\sin i \end{aligned}$$

and again

$$\begin{aligned}
\mathbf{k} &= k\sin i\hat{\mathbf{i}} + k\cos i\hat{\mathbf{k}} \\
\mathbf{k}' &= k'\sin r\hat{\mathbf{i}} + k'\cos r\hat{\mathbf{k}} \\
\mathbf{k}'' &= k\sin i\hat{\mathbf{i}} - k\cos i\hat{\mathbf{k}}
\end{aligned}$$

we substitute into the boundary conditions, one at a time. For the first,

$$\begin{aligned}
\epsilon\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\cdot\hat{\mathbf{k}} &= \epsilon'\boldsymbol{\mathcal{E}}'\cdot\hat{\mathbf{k}} \\
\epsilon\left(-\hat{\mathbf{i}}\mathcal{E}\cos i + \hat{\mathbf{k}}\mathcal{E}\sin i + \hat{\mathbf{i}}\mathcal{E}''\cos i + \hat{\mathbf{k}}\mathcal{E}''\sin i\right)\cdot\hat{\mathbf{k}} &= \epsilon'\left(-\hat{\mathbf{i}}\mathcal{E}'\cos r + \hat{\mathbf{k}}\mathcal{E}'\sin r\right)\cdot\hat{\mathbf{k}} \\
\epsilon\mathcal{E}\sin i + \epsilon\mathcal{E}''\sin i &= \epsilon'\mathcal{E}'\sin r
\end{aligned}$$

while the second becomes

$$\begin{aligned}
\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\times\hat{\mathbf{k}} &= \boldsymbol{\mathcal{E}}'\times\hat{\mathbf{k}} \\
\left(-\hat{\mathbf{i}}\mathcal{E}\cos i + \hat{\mathbf{k}}\mathcal{E}\sin i + \hat{\mathbf{i}}\mathcal{E}''\cos i + \hat{\mathbf{k}}\mathcal{E}''\sin i\right)\times\hat{\mathbf{k}} &= \left(-\hat{\mathbf{i}}\mathcal{E}'\cos r + \hat{\mathbf{k}}\mathcal{E}'\sin r\right)\times\hat{\mathbf{k}} \\
\mathcal{E}\cos i - \mathcal{E}''\cos i &= \mathcal{E}'\cos r
\end{aligned}$$

The third is identically satisfied,

$$\begin{aligned}
\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\cdot\hat{\mathbf{k}} &= \left(\mathbf{k}'\times\boldsymbol{\mathcal{E}}'\right)\cdot\hat{\mathbf{k}} \\
\left(\left(k\sin i\hat{\mathbf{i}} + k\cos i\hat{\mathbf{k}}\right)\times\left(-\hat{\mathbf{i}}\mathcal{E}\cos i + \hat{\mathbf{k}}\mathcal{E}\sin i\right)\right)\cdot\hat{\mathbf{k}} & \\
+\left(\left(k\sin i\hat{\mathbf{i}} - k\cos i\hat{\mathbf{k}}\right)\times\left(\hat{\mathbf{i}}\mathcal{E}''\cos i + \hat{\mathbf{k}}\mathcal{E}''\sin i\right)\right)\cdot\hat{\mathbf{k}} &= \left(\left(k'\sin r\hat{\mathbf{i}} + k'\cos r\hat{\mathbf{k}}\right)\times\left(-\hat{\mathbf{i}}\mathcal{E}'\cos r + \hat{\mathbf{k}}\mathcal{E}'\sin r\right)\right)\cdot\hat{\mathbf{k}} \\
\left(-k\sin i\mathcal{E}\cos i + k\sin i\mathcal{E}''\cos i\right)\left(\hat{\mathbf{i}}\times\hat{\mathbf{i}}\right)\cdot\hat{\mathbf{k}} &= -\left(k'\sin r\mathcal{E}'\cos r\hat{\mathbf{i}}\times\hat{\mathbf{i}}\right)\cdot\hat{\mathbf{k}} \\
0 &= 0
\end{aligned}$$

and, finally, the fourth gives

$$\begin{aligned}
\frac{1}{\mu}\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\times\hat{\mathbf{k}} &= \frac{1}{\mu'}\left(\mathbf{k}'\times\boldsymbol{\mathcal{E}}'\right)\times\hat{\mathbf{k}} \\
\frac{1}{\mu}\left(\left(k\sin i\hat{\mathbf{i}} + k\cos i\hat{\mathbf{k}}\right)\times\left(-\hat{\mathbf{i}}\mathcal{E}\cos i + \hat{\mathbf{k}}\mathcal{E}\sin i\right)\right)\times\hat{\mathbf{k}} & \\
+\frac{1}{\mu}\left(\left(k\sin i\hat{\mathbf{i}} - k\cos i\hat{\mathbf{k}}\right)\times\left(\hat{\mathbf{i}}\mathcal{E}''\cos i + \hat{\mathbf{k}}\mathcal{E}''\sin i\right)\right)\times\hat{\mathbf{k}} &= \frac{1}{\mu'}\left(\left(k'\sin r\hat{\mathbf{i}} + k'\cos r\hat{\mathbf{k}}\right)\times\left(-\hat{\mathbf{i}}\mathcal{E}'\cos r + \hat{\mathbf{k}}\mathcal{E}'\sin r\right)\right)\times\hat{\mathbf{k}} \\
\left(\frac{1}{\mu}k\mathcal{E}\sin^2 i + \frac{1}{\mu}k\mathcal{E}\cos^2 i\right)\left(\hat{\mathbf{i}}\times\hat{\mathbf{k}}\right)\times\hat{\mathbf{k}} & \\
+\left(\frac{1}{\mu}k\mathcal{E}''\sin^2 i + \frac{1}{\mu}k\mathcal{E}''\cos^2 i\right)\left(\hat{\mathbf{i}}\times\hat{\mathbf{k}}\right)\times\hat{\mathbf{k}} &= \left(\frac{1}{\mu'}k'\mathcal{E}'\cos^2 r + \frac{1}{\mu'}k'\mathcal{E}'\sin^2 r\right)\left(\hat{\mathbf{i}}\times\hat{\mathbf{k}}\right)\times\hat{\mathbf{k}} \\
\frac{1}{\mu}k\left(\mathcal{E} + \mathcal{E}'\right) &= \frac{1}{\mu'}k'\mathcal{E}' \\
\frac{\omega}{c}\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E} + \mathcal{E}'\right) &= \frac{\omega}{c}\sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}' \\
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E} + \mathcal{E}'\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

Collecting the results,

$$\begin{aligned}
\epsilon\left(\mathcal{E}+\mathcal{E}''\right)\sin i &= \epsilon'\mathcal{E}'\sin r \\
\mathcal{E}\cos i-\mathcal{E}''\cos i &= \mathcal{E}'\cos r \\
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}+\mathcal{E}'\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

we see we can use Snell's law on the first. Substituting $\frac{\epsilon}{k}=\frac{c}{\omega}\sqrt{\frac{\epsilon}{\mu}}$,

$$\begin{aligned}
\frac{\epsilon}{k}\left(\mathcal{E}+\mathcal{E}''\right) &= \frac{\epsilon'}{k'}\mathcal{E}' \\
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}+\mathcal{E}''\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

showing that it reproduces the fourth. Either lets us solve for $\mathcal{E}'$,

$$\mathcal{E}'=\sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\left(\mathcal{E}+\mathcal{E}''\right)$$

and substitute into the remaining equation and solve for $\frac{\mathcal{E}''}{\mathcal{E}}$,

$$\begin{aligned}
\mathcal{E}\cos i-\mathcal{E}''\cos i &= \sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\left(\mathcal{E}+\mathcal{E}''\right)\cos r \\
\mathcal{E}''\cos i+\sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\mathcal{E}''\cos r &= \mathcal{E}\cos i-\sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\mathcal{E}\cos r \\
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{\cos i-\sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\cos r}{\cos i+\sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\cos r}
\end{aligned}$$

Finally, rewrite $\sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}=\frac{n\mu'}{n'\mu}$ and express $\cos r$ in terms of the angle of incidence

$$\begin{aligned}
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{\cos i-\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\cos r}{\cos i+\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\cos r} \\
&= \frac{\cos i-\frac{n\mu'}{n'\mu}\frac{1}{n'}\sqrt{n'^2-n^2\sin^2 i}}{\cos i+\frac{n\mu'}{n'\mu}\frac{1}{n'}\sqrt{n'^2-n^2\sin^2 i}} \\
&= \frac{n'^2\cos i-\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i+\frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}
\end{aligned}$$

Now we find the amplitude $\frac{\mathcal{E}'}{\mathcal{E}}$ of the refracted wave. Substituting into $\mathcal{E}'=\sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\left(\mathcal{E}+\mathcal{E}''\right)$,

$$\frac{\mathcal{E}'}{\mathcal{E}} = \sqrt{\frac{\epsilon\mu'}{\mu\epsilon'}}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right)$$

$$\begin{aligned}
&= \frac{n\mu'}{n'\mu}\left(1+\frac{n'^2\cos i - \frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}\right)\\
&= \frac{n\mu'}{n'\mu}\left(\frac{2n'^2\cos i}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}\right)\\
&= \frac{2nn'\frac{\mu'}{\mu}\cos i}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}\\
&= \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i}}
\end{aligned}$$

Therefore, the electric field amplitudes for incidence with polarization in the plane of the scattering are given by

$$\begin{aligned}
\frac{\mathcal{E}'}{\mathcal{E}} &= \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i}}\\
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{n'^2\cos i - \frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2-n^2\sin^2 i}}
\end{aligned}$$

Special cases

There are two special cases. For simplicity, we set $\mu = \mu'$.

Total internal reflection

For polarization perpendicular to the plane of incidence, the direction of the refracted wave is given by Snell's law,

$$n\sin i = n'\sin r$$

and r becomes $\frac{\pi}{2}$ if $n > n'$ and

$$\sin i = \frac{n'}{n}$$

This means that the refracted wave moves only in the x-direction, and not into the second medium at all. We have total internal reflection, and the wave stays in the first medium.

Total absorption

For polarization parallel to the plane of incidence, it is possible for the reflected wave to vanish. With $\mu = \mu'$, the conditon for this is

$$\begin{aligned}
0 &= \frac{\mathcal{E}''}{\mathcal{E}}\\
&= \frac{n'^2\cos i - n\sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i}}\\
n'^2\cos i &= n\sqrt{n'^2-n^2\sin^2 i}\\
n'^4\cos^2 i &= n^2n'^2 - n^4\sin^2 i\\
n'^4 - n'^4\sin^2 i &= n^2n'^2 - n^4\sin^2 i\\
n'^4 - n^2n'^2 &= \left(n'^4 - n^4\right)\sin^2 i
\end{aligned}$$

$$
\begin{aligned}
\left(n'^2 - n^2\right) n'^2 &= \left(n'^2 - n^2\right)\left(n'^2 + n^2\right)\sin^2 i \\
\sin i &= \frac{n'}{\sqrt{n'^2 + n^2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\cos i &= \sqrt{1 - \sin^2 i} \\
&= \sqrt{1 - \frac{n'^2}{n'^2 + n^2}} \\
&= \frac{n}{\sqrt{n'^2 + n^2}}
\end{aligned}
$$

and therefore,

$$
\tan i = \frac{n'}{n}
$$

This angle, $i_B = \tan^{-1} \frac{n'}{n}$, is called Brewster's angle.

This cannot happen for the polarization perpendicular to the plane of incidence, since this would require

$$
\begin{aligned}
0 &= \frac{n \cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}}{n \cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}} \\
n \cos i &= \sqrt{n'^2 - n^2 \sin^2 i} \\
n^2 \cos^2 i &= n'^2 - n^2 \sin^2 i \\
n &= n'
\end{aligned}
$$

Therefore, when light with a mixture of polarizations strikes a surface at or near Brewster's angle, only the polarization perpendicular to the plane of incidence is reflected. This is the reason sunglasses are so effect while driving. Light reflected from the pavement can be partially or fully polarized, so polarized lenses can cut out almost all the reflected light.