Plane electromagnetic waves

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We start with the Maxwell equations in a uniform medium without sources:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= 0\\
\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\
\boldsymbol{\nabla}\times\mathbf{B}-\frac{\mu\epsilon}{c}\frac{\partial\mathbf{E}}{\partial t} &= 0
\end{aligned}$$

Since we know the solutions must satisfy the wave equations,

$$\begin{aligned}
\left(-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}+\bigtriangledown^2\right)\mathbf{E} &= 0\\
\left(-\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}+\bigtriangledown^2\right)\mathbf{B} &= 0
\end{aligned}$$

where we assume linearity of the medium, we may seek plane wave solutions and build more complicated waves by superposition. We set

$$\begin{aligned}
\mathbf{E} &= \boldsymbol{\mathcal{E}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\\
\mathbf{B} &= \boldsymbol{\mathcal{B}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}
\end{aligned}$$

with the same dependence for the other fields. Notice that we have made an analytic extension to complex fields. Because the Maxwell equations are linear, either the real or the imaginary parts will solve the equations. It is also useful to allow ϵ and μ be complex functions of frequency for the description of dissipation and dispersion. For now, think of ϵ and μ as real.

Substituting into the wave equations shows that we must have

$$\frac{\mu\epsilon}{c^2}\omega^2-\mathbf{k}^2 = 0$$

so that $\frac{\sqrt{\mu\epsilon}}{c}\omega=k=|\mathbf{k}|$. This solves the wave equations, but does not completely solve all of the Maxwell equations. Substituting the plane wave form into the Maxwell equations, the space and time dependence drops out, leaving constraints on the coefficient vectors. Writing the wave vector $\mathbf{k}$ as $\mathbf{k}=k\hat{\mathbf{n}}$,

$$\begin{aligned}
\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{E}} &= 0\\
\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{B}} &= 0\\
k\hat{\mathbf{n}}\times\boldsymbol{\mathcal{E}}-\frac{\omega}{c}\boldsymbol{\mathcal{B}} &= 0\\
k\hat{\mathbf{n}}\times\boldsymbol{\mathcal{B}}+\frac{\mu\epsilon\omega}{c}\boldsymbol{\mathcal{E}} &= 0
\end{aligned}$$

Notice that since the cross products with $\hat{\mathbf{n}}$ are orthogonal to $\hat{\mathbf{n}}$, the first two equations follow automatically from the second pair of equations. From the third,

$$\begin{aligned}\boldsymbol{\mathcal{B}} &= \frac{kc}{\omega}\hat{\mathbf{n}} \times \boldsymbol{\mathcal{E}} \\ &= \sqrt{\mu\epsilon}\hat{\mathbf{n}} \times \boldsymbol{\mathcal{E}}\end{aligned}$$

Substituting this into the final equation,

$$\begin{aligned}0 &= k\hat{\mathbf{n}} \times \boldsymbol{\mathcal{B}} + \frac{\mu\epsilon\omega}{c}\boldsymbol{\mathcal{E}} \\ &= \sqrt{\mu\epsilon}\frac{\sqrt{\mu\epsilon}}{c}\omega\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \boldsymbol{\mathcal{E}} + \frac{\mu\epsilon\omega}{c}\boldsymbol{\mathcal{E}} \\ &= -\frac{\mu\epsilon\omega}{c}\boldsymbol{\mathcal{E}} + \frac{\mu\epsilon\omega}{c}\boldsymbol{\mathcal{E}}\end{aligned}$$

we find it is identically satisfied.

A point of constant phase φ_0 of the wave is now given by

$$\begin{aligned}\varphi_0 &= \mathbf{k} \cdot \mathbf{x} - \omega t \\ &= k\left(\hat{\mathbf{n}} \cdot \mathbf{x} \pm \frac{c}{\sqrt{\mu\epsilon}}t\right)\end{aligned}$$

For example, let the wave propagate in the x-direction so that $\hat{\mathbf{n}} = \hat{\mathbf{i}}$. Then for a right-moving wave,

$$x = \frac{\varphi_0}{k} + \frac{c}{\sqrt{\mu\epsilon}}t$$

so that point (for example, the crest of the wave) moves with velocity

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}}$$

This is the speed of the wave in the medium. It is called the phase velocity.

We define the index of refraction, n, as the ratio of the speed of light in vacuum to the speed of light in a medium:

$$\begin{aligned}n &\equiv \frac{c}{v} \\ &= \sqrt{\mu\epsilon}\end{aligned}$$

For a plane electromagnetic wave, with real index of refraction, we therefore have the electric field, magnetic field, and direction of propagation all mutually perpendicular. The magnitude of the magnetic field is related to the magnitude of the electric field by

$$\mathcal{B} = n\mathcal{E}$$

To completely specify the wave we therefore need:

1. The frequency, ω.
2. The index of refraction, n
3. The direction of propagation, $\hat{\mathbf{n}}$.
4. The magnitude and direction of $\boldsymbol{\mathcal{E}}$ in the plane perpendicular to $\hat{\mathbf{n}}$.

To specify the direction of the electric field, it is useful to introduce a set of orthonormal basis vectors, $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \mathbf{n}$, with $\boldsymbol{\varepsilon}_1$ or $\boldsymbol{\varepsilon}_2$ giving the direction of $\boldsymbol{\mathcal{E}}$ and with $\mathbf{n} \times \boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2$ or $\mathbf{n} \times \boldsymbol{\varepsilon}_2 = -\boldsymbol{\varepsilon}_1$ giving the corresponding direction of $\boldsymbol{\mathcal{B}}$. The direction of the electric field is called the polarization. A general wave may be a superposition of different frequencies, different directions $\mathbf{n}$, and/or different polarizations.

We may also find the Poynting vector complexified. To get the right expression, consider the real part of our solution above,

$$\begin{aligned}
\mathbf{E} &= \boldsymbol{\varepsilon}_1 \mathcal{E} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \\
\mathbf{B} &= \boldsymbol{\mathcal{B}} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \\
&= \boldsymbol{\varepsilon}_2 n \mathcal{E} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)
\end{aligned}$$

For these, the Poynting vector is

$$\begin{aligned}
\mathbf{S} &= \mathbf{E} \times \mathbf{H} \\
&= \boldsymbol{\varepsilon}_1 \mathcal{E} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \times \frac{1}{\mu} \boldsymbol{\varepsilon}_2 n \mathcal{E} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \\
&= \boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2 \frac{n}{\mu} \mathcal{E}^2 \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t) \\
&= \hat{\mathbf{n}} \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t)
\end{aligned}$$

so the time average is

$$\begin{aligned}
\langle \mathbf{S} \rangle &= \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n} \left\langle \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t) \right\rangle \\
&= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n}
\end{aligned}$$

The momentum flux is in the direction of travel of the wave, with magnitude proportional to $\mathbf{E}^* \mathbf{E} = \mathcal{E}^2$.

This result differs from Jackson's result by a factor of $\frac{1}{2}$. However, Jackson states that the time averaged flux of energy is given by the real part of

$$\begin{aligned}
\mathbf{S} &= \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \\
&= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n}
\end{aligned}$$

This agrees with our example. What happens is that the factor of $\frac{1}{2}$ gives the time average of a sine or cosine wave, while the complex conjugation cancels the phase factors altogether. However, eq. 7.13 cannot be correct because it has no time dependence, whereas a real wave will have oscillating flux,

$$\mathbf{S} = \sqrt{\frac{\epsilon}{\mu}} \mathcal{E}^2 \mathbf{n} \cos^2(\mathbf{k} \cdot \mathbf{x} - \omega t)$$