The Maxwell equations

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1 The time-dependent Maxwell equations

We have shown that the static form of the Maxwell equations for electrostatics are

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{D} &= \rho \\ \boldsymbol{\nabla} \times \mathbf{E} &= 0 \end{aligned}$$

In formulating the second of these, we ignored the possible presence of changing magnetic fields. In general, we know that a changing magnetic flux produces a contrary $\mathcal{EMF}$ according to Lenz's law,

$$\oint\limits_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int\limits_S \mathbf{B} \cdot \mathbf{n} d^2 x$$

where C is the boundary curve of the area S. Employing Stokes' theorem, and bringing the time derivative inside the integral,

$$\begin{aligned} \int\limits_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{n} d^2 x &= -\int\limits_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} d^2 x \\ \int\limits_S \left(\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{n} d^2 x &= 0 \end{aligned}$$

Since the surface S is arbitrary, we must have

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

in the non-static case.

For magnetostatics, we have

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{H} &= \mu_0 \mathbf{J} \end{aligned}$$

where we have assumed the the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

reduces to just the vanishing divergence of the current, $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$. We can find the full equations by extending Ampère's law to make it consistent with the general form of the continuity equation. Consider the divergence of Ampère's law. The left side gives

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) = 0$$

but this is not consistent because the right side gives

$$\begin{aligned}
\mu_0 \boldsymbol{\nabla}\cdot\mathbf{J} &= -\mu_0\frac{\partial\rho}{\partial t}\\
&= -\mu_0\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right)\\
&= -\mu_0\boldsymbol{\nabla}\cdot\frac{\partial\mathbf{D}}{\partial t}
\end{aligned}$$

The equation is consistent if we include $-\mu_0\frac{\partial\mathbf{D}}{\partial t}$ on the left side, so that

$$\boldsymbol{\nabla}\times\mathbf{H}-\mu_0\frac{\partial\mathbf{D}}{\partial t}=\mu_0\mathbf{J}$$

These two modifications to the static equations bring us to the full Maxwell equations.

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon}\\
\boldsymbol{\nabla}\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\
\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \mu\mathbf{J}
\end{aligned}$$

It is convenient to use Gaussian units for the field so that $\epsilon_0=\frac{1}{4\pi c}$ and $\mu_0=\frac{4\pi}{c}$, and the electric field is scaled by c, so that $\frac{1}{c}\mathbf{E}\rightarrow\mathbf{E}$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{4\pi\rho}{\epsilon} && (1)\\
\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0 && (2)\\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0 && (3)\\
\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi\mu}{c}\mathbf{J} && (4)
\end{aligned}$$

where ϵ and μ go to the relative permitivity and permeability, $\epsilon\rightarrow\frac{\epsilon}{\epsilon_0}$ and $\mu\rightarrow\frac{\mu}{\mu_0}$. This is the general macroscopic form in Gaussian units.

2 Wave equations for the fields

Taking the curl of eq.(2), and substiting for the curl of the magnetic field using Ampère's law,

$$\begin{aligned}
\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right) &= 0\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})+\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi\mu}{c}\mathbf{J}\right) &= 0\\
\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{E}\right)-\nabla^2\mathbf{E}+\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{4\pi\mu}{c^2}\frac{\partial\mathbf{J}}{\partial t} &= 0
\end{aligned}$$

so that using Gauss's law,

$$-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}+\nabla^2\mathbf{E}=\frac{4\pi}{\epsilon}\left(\boldsymbol{\nabla}\rho+\frac{\partial\mathbf{J}}{\partial t}\right)$$

Defining the d'Alembertian wave operator,

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

gives the wave equation for the electric field,

$$\Box \mathbf{E} = \frac{4\pi}{\epsilon}\left(\boldsymbol{\nabla}\rho + \frac{\partial \mathbf{J}}{\partial t}\right)$$

For the magnetic field, take the curl of eq.(4) and use Faraday's law to eliminate the electric field,

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) - \frac{1}{c}\boldsymbol{\nabla} \times \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi\mu}{c}\boldsymbol{\nabla} \times \mathbf{J} \\
\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \nabla^2 \mathbf{B} + \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= \frac{4\pi\mu}{c}\boldsymbol{\nabla} \times \mathbf{J}
\end{aligned}$$

With the vanishing divergence from Gauss's law for magnetism, we have

$$\Box \mathbf{B} = -\frac{4\pi\mu}{c}\boldsymbol{\nabla} \times \mathbf{J}$$

In the absence of sources we have simple d'Alembertian waves,

$$\begin{aligned}
\Box \mathbf{E} &= 0 \\
\Box \mathbf{B} &= 0
\end{aligned}$$

Clearly, these will propagate with the speed of light.

3 Helmholz theorem

Given an arbitrary differentiable vector field, $\mathbf{v}$, in a region V satisfying given boundary conditions $\mathbf{v}(S)$. Write

$$\mathbf{v} = \boldsymbol{\nabla} f + \mathbf{w}$$

where we are free to choose f. Taking the divergence, we have

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \nabla^2 f + \boldsymbol{\nabla} \cdot \mathbf{w}$$

Given $\mathbf{v}$, its divergence is some function g and we choose f to satisfy

$$\nabla^2 f = g$$

with boundary condition $\nabla^2 f(S) = g(S) = \boldsymbol{\nabla} \cdot \mathbf{v}(S)$, so that $\mathbf{w}$ is divergence-free, $\boldsymbol{\nabla} \cdot \mathbf{w} = 0$.

Now, let

$$\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u} + \mathbf{s}$$

with $\mathbf{u}$ determined up to a gradient. Choose the gradient so $\mathbf{u}$ is divergence free. Then

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{w} &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) + \boldsymbol{\nabla} \cdot \mathbf{s} \\
\boldsymbol{\nabla} \cdot \mathbf{s} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{w} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}) + \boldsymbol{\nabla} \times \mathbf{s} \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2 \mathbf{u} + \boldsymbol{\nabla} \times \mathbf{s} \\
&= -\nabla^2 \mathbf{u} + \boldsymbol{\nabla} \times \mathbf{s}
\end{aligned}$$

or

$$\boldsymbol{\nabla} \times \mathbf{s} = \boldsymbol{\nabla} \times \mathbf{w} + \nabla^2 \mathbf{u}$$

Let the known vector field

$$\mathbf{j} = \boldsymbol{\nabla} \times \mathbf{w} = \boldsymbol{\nabla} \times \mathbf{v}$$

be the source for solving

$$\nabla^2 \mathbf{u} = -\mathbf{j}$$

Then

$$\boldsymbol{\nabla} \times \mathbf{s} = 0$$

Taking another curl of this, $\mathbf{s}$ satisfies the Laplace equation,

$$\begin{aligned} 0 &= \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{s}) - \nabla^2 \mathbf{s} \\ &= -\nabla^2 \mathbf{s} \end{aligned}$$

Therefore, our arbitrary vector field $\mathbf{v}$ may be written as a gradient plus a curl plus a solution to the Laplace equation:

$$\mathbf{v} = \boldsymbol{\nabla} f + \boldsymbol{\nabla} \times \mathbf{u} + \mathbf{s}$$

We may choose $\mathbf{s}$ so that $\mathbf{v}$ satisfies the correct boundary conditions. This is the Helmholz theorem.

Suppose $\mathbf{v}$ is define on all of R^3 and vanishes at infinity. Then we may choose $f, \mathbf{u}$ and $\mathbf{s}$ to vanish at infinity. Since $\mathbf{s}$ satisfies the Laplace equation and vanishes at infinity, it must vanish everywhere, so that $\mathbf{v}$ may be written as

$$\mathbf{v} = \boldsymbol{\nabla} f + \boldsymbol{\nabla} \times \mathbf{u}$$

We also have this condition if we can choose f and $\mathbf{u}$ to completely satisfy the boundary conditions on $\mathbf{v}$, so that $\mathbf{s}$vanishes on the boundary and therefore throughout the region. This holds whenever we have an isolated system.

4 Potentials

4.1 The existence of the scalar and vector potentials

The Maxwell equations are of two types, homogeneous and inhomogeneous. The two homogeneous equations,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \end{aligned}$$

imply the existence of scalar and vector potentials.

Suppose we have an isolated system. Then we may write $\mathbf{B}$ using the Helmholz theorem with $\mathbf{s} = 0$,

$$\mathbf{B} = \boldsymbol{\nabla} f + \boldsymbol{\nabla} \times \mathbf{A}$$

Taking the divergence leaves us with

$$\nabla^2 f = 0$$

and since the system is isolated the fields vanish at infinity and f must be zero. Therefore, there exists a vector field $\mathbf{A}$ such that

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

Any $\mathbf{A}'$ which differs from $\mathbf{A}$ by a gradient also satisfies $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}'$.

Now consider the second homogeneous equation,

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

Substituting for $\mathbf{B}$ this becomes

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

Again using the Helmholtz theorem, this time for $\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$,

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \boldsymbol{\nabla}(-\Phi) + \boldsymbol{\nabla} \times \mathbf{u}$$

The vanishing curl of the left hand side and the gradient shows that

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}) \\ &= -\nabla^2 \mathbf{u} \end{aligned}$$

where we have used our choice of $\mathbf{u}$ as divergence free. Since, in addition, $\mathbf{u}$ vanishes on the boundary, we have $\mathbf{u} = 0$ everywhere. This gives us an expression for the electric field in terms of the scalar and vector potentials,

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Notice that neither potential is unique so these potentials are not physical. Nonetheless, they give a simpler path to solutions to Maxwell's equations than working with the fields directly.

Since the curl of a gradient vanishes, we may add any gradient to the vector potential $\mathbf{A}$ without changing the magnetic field. In order for this transformation to keep the electric field unchanged we must also change Φ. Let

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} f$$

for any function f. Then invariance of the electric field requires

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\Phi' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t} \\ &= -\boldsymbol{\nabla}\Phi' - \frac{1}{c}\frac{\partial(\mathbf{A} + \boldsymbol{\nabla} f)}{\partial t} \\ &= -\boldsymbol{\nabla}\left(\Phi' + \frac{1}{c}\frac{\partial f}{\partial t} \right) - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \end{aligned}$$

and therefore

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

The simultaneous transformations

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \boldsymbol{\nabla} f \\ \Phi' &= \Phi - \frac{1}{c}\frac{\partial f}{\partial t} \end{aligned}$$

leave both fields unchanged.

This freedom allows us to place convenient constraints on the potentials. For example, suppose we have a general pair of potentials $\tilde{\Phi}, \tilde{\mathbf{A}}$. Perform a gauge transformation,

$$\begin{aligned} \mathbf{A} &= \tilde{\mathbf{A}} + \boldsymbol{\nabla} f \\ \Phi &= \tilde{\Phi} - \frac{1}{c}\frac{\partial f}{\partial t} \end{aligned}$$

and consider the combination

$$\begin{aligned}\frac{1}{c}\frac{\partial\Phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{c}\frac{\partial}{\partial t}\left(\tilde{\Phi}-\frac{1}{c}\frac{\partial f}{\partial t}\right)+\boldsymbol{\nabla}\cdot\left(\tilde{\mathbf{A}}+\boldsymbol{\nabla}f\right)\\ &= \left(\frac{1}{c}\frac{\partial\tilde{\Phi}}{\partial t}+\boldsymbol{\nabla}\cdot\tilde{\mathbf{A}}\right)-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}+\nabla^2 f\end{aligned}$$

The combination $g=\frac{1}{c}\frac{\partial\tilde{\Phi}}{\partial t}+\boldsymbol{\nabla}\cdot\tilde{\mathbf{A}}$ is some definite function and we may choose the *Lorentz gauge*,

$$\frac{1}{c}\frac{\partial\Phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}=0$$

by letting f solve

$$-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}+\nabla^2 f=g\left(\mathbf{x},t\right)$$

We use the Lorentz gauge below. Notice that this choice for f does not exhaust all possible gauge transformations, since any further transformation by f' satisfying

$$-\frac{1}{c^2}\frac{\partial^2 f'}{\partial t^2}+\nabla^2 f'=0$$

leaves the Lorentz condition unaltered and allows us to constrain the potentials further.

4.2 Wave equations for the potentials

Now consider the inhomogeneous equations, rewritten in terms of the potentials. Choosing Gaussian units for the fields, the equations are

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\frac{\rho}{\epsilon}\\ \boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mu\mathbf{J}\end{aligned}$$

For Gauss's law

$$\begin{aligned}-\nabla^2\Phi-\frac{1}{c}\boldsymbol{\nabla}\cdot\frac{\partial\mathbf{A}}{\partial t} &= 4\pi\frac{\rho}{\epsilon}\\ -\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}+\nabla^2\Phi+\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\Phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}\right) &= -4\pi\frac{\rho}{\epsilon}\end{aligned}$$

while for Ampère's law,

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mu\mathbf{J}\\ \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right)-\frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\Phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) &= \frac{4\pi}{c}\mu\mathbf{J}\\ \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)-\nabla^2\mathbf{A}+\frac{1}{c}\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= \frac{4\pi}{c}\mu\mathbf{J}\\ -\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla^2\mathbf{A}-\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}\right) &= -\frac{4\pi}{c}\mu\mathbf{J}\end{aligned}$$

These equations simplify if we choose the Lorentz gauge. In terms of the d'Alembertian and the Lorentz gauge, we have the wave equations,

$$\begin{aligned} \Box\Phi &= -4\pi\frac{\rho}{\epsilon} \\ \Box\mathbf{A} &= -\frac{4\pi}{c}\mu\mathbf{J} \end{aligned}$$