Green Functions

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1 The Laplacian of $\frac{1}{r}$ and the Dirac delta function

Consider the potential of an isolated point charge q at $\mathbf{x}'$

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

For convenience, choose coordinates so that $\mathbf{x}'$ is at the origin. Then in spherical coordinates, the potential is proportional to $\frac{1}{r}$. As a function, $f = \frac{1}{r}$ is defined on the *open* interval $(0,\infty)$, but not at the origin. Its Laplacian is also defined on this interval, and is quickly seen to vanish everywhere,

$$\begin{aligned}
\nabla^2\left(\frac{1}{r}\right) &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right) \\
&= \frac{1}{r^2}\frac{d}{dr}(-1) \\
&= 0
\end{aligned}$$

This leads to a difficulty when we consider the divergence theorem, for which the volume integral includes the origin

$$\int\limits_V \nabla^2\left(\frac{1}{r}\right)d^3x = \oint\limits_S \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right)d^2x$$

since the right hand side is well-defined but the left is not. Indeed, for a sphere S_ε, of radius ε, the integral on the right becomes

$$\begin{aligned}
\oint\limits_{S_\varepsilon} \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right)d^2x &= \int\limits_0^\pi\int\limits_0^{2\pi}\hat{\mathbf{r}}\cdot\left(-\hat{\mathbf{r}}\frac{1}{\varepsilon^2}\right)\varepsilon^2\sin\theta d\theta d\varphi \\
&= -4\pi
\end{aligned}$$

However, the integral on the left is undefined.

The rigorous way to handle this is to *extend* the *function* $f = \frac{1}{r}$ to a *distribution.* A distribution is defined as the limit of a sequence of functions, giving an object which is only meaningful when integrated. Thus, if we define a distribution f to be the limit

$$f(x) \equiv \lim_{a\to 0} f_a(x)$$

where $f_a(x)$ is a collection of functions depending on a parameter a. A distribution is often called a *functional*, and we use the two terms interchangeably.

The integral of the distribution is defined as the limit of the well-behaved integrals of the series of functions

$$\int f(x)\,dx \equiv \lim_{a\to 0}\int f_a(x)\,dx$$

and this may be perfectly finite even if $f(x)$ is not a true function.

With this in mind, let $f_a(\mathbf{x}) = \frac{1}{\sqrt{r^2+a^2}}$. This is defined for the closed interval $r \in [0,\infty]$, and so is its Laplacian

$$\begin{aligned}
\nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right) &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\frac{1}{\sqrt{r^2+a^2}}\right) \\
&= \frac{1}{r^2}\frac{d}{dr}\left(-\frac{1}{2}r^2\frac{2r}{(r^2+a^2)^{3/2}}\right) \\
&= -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^3}{(r^2+a^2)^{3/2}}\right) \\
&= -\frac{1}{r^2}\left(\frac{3r^2}{(r^2+a^2)^{3/2}} - \frac{3}{2}\frac{2r^4}{(r^2+a^2)^{5/2}}\right) \\
&= -\frac{3}{(r^2+a^2)^{3/2}} + \frac{3r^2}{(r^2+a^2)^{5/2}} \\
&= \frac{3r^2}{(r^2+a^2)^{5/2}} - \frac{3\left(r^2+a^2\right)}{(r^2+a^2)^{5/2}} \\
&= -\frac{3a^2}{(r^2+a^2)^{5/2}}
\end{aligned}$$

We may therefore define a distribution to extend $\delta(\mathbf{x}) = \nabla^2\left(\frac{1}{r}\right)$ by

$$\begin{aligned}
\delta(\mathbf{x}) &= \lim_{a\to 0} f_a(x) \\
&= \lim_{a\to 0}\left(-\frac{3a^2}{(r^2+a^2)^{5/2}}\right)
\end{aligned}$$

The integral is now well-defined:

$$\begin{aligned}
\int\limits_V \nabla^2\left(\frac{1}{r}\right) d^3x &\equiv \lim_{a\to 0}\int\limits_V \nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right) d^3x \\
&= -12\pi \lim_{a\to 0} a^2 \int\limits_0^{\varepsilon} \frac{r^2 dr}{(r^2+a^2)^{5/2}} \\
&= -4\pi \lim_{a\to 0}\int\limits_0^{\varepsilon} \frac{d}{dr}\frac{r^3}{(r^2+a^2)^{3/2}} dr \\
&= -\lim_{a\to 0}\frac{4\pi\varepsilon^3}{(\varepsilon^2+a^2)^{3/2}} \\
&= -4\pi
\end{aligned}$$

for any finite ε. As a pleasant bonus, the divergence theorem is now satisfied as long as we understand $\nabla^2\left(\frac{1}{r}\right)$ to be a distribution.

Notice that the functional $\delta(r)$ diverges at $r=0$ and vanishes for all $r>0$, while its integral is finite. Furthermore, if $f(r)$ is any smooth, bounded function of r whih vanishes outside some compact set, then

$$\begin{aligned}\int_V f(r)\nabla^2\left(\frac{1}{r}\right)d^3x &= \lim_{a\to 0}\int_V f(r)\nabla^2\left(\frac{1}{\sqrt{r^2+a^2}}\right)d^3x \\ &= -4\pi f(0)\lim_{a\to 0}\int_0^{\varepsilon}\frac{d}{dr}\frac{r^3}{(r^2+a^2)^{3/2}}dr - 4\pi\lim_{a\to 0}\int_0^{\varepsilon} r\frac{df}{dr}\frac{d}{dr}\frac{r^3}{(r^2+a^2)^{3/2}}dr+\ldots \\ &= -4\pi f(0) - 4\pi R\end{aligned}$$

where the remainder R satisfies

$$\begin{aligned}R &= \lim_{a\to 0}\int_0^{\varepsilon} r\frac{df}{dr}\frac{d}{dr}\frac{r^3}{(r^2+a^2)^{3/2}}dr+\ldots \\ &< \varepsilon f'(0)\lim_{a\to 0}\int_0^{\varepsilon}\frac{d}{dr}\frac{r^3}{(r^2+a^2)^{3/2}}dr+\ldots\end{aligned}$$

which vanishes as $\varepsilon\to 0$, leaving

$$\int_V f(r)\nabla^2\left(\frac{1}{r}\right)d^3x = -4\pi f(0)$$

This means that $\nabla^2\left(\frac{1}{r}\right)=\nabla^2\left(\frac{1}{|\mathbf{x}|}\right)$ is proportional to a Dirac delta function(al) at the origin,

$$\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = -4\pi\delta^3(\mathbf{x})$$

Returning to an arbitrary origin, we may write this as

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

Notice that what is universally referred to as the Dirac delta function is, properly speaking, a functional.

2 General solution for the potential of a point charge with boundary conditions

In terms of the potential, Gauss's law in free space is

$$\nabla^2\Phi = -\frac{1}{\epsilon_0}\rho(\mathbf{x})$$

The charge density for an isolated charge q at position $\mathbf{x}'$ is

$$\rho(\mathbf{x}) = q\delta^3(\mathbf{x}-\mathbf{x}')$$

We wish to solve for the potential Φ for this point source, and boundary conditions given on some surface, S. The boundary may be comprised of multiple pieces.

From the preceding section, we see that the solution for the potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{x}-\mathbf{x}'|}+\Phi_0$$

where Φ_0 is any solution to the Laplace equation,

$$\nabla^2 \Phi_0 = 0$$

Now, we know that the solution to the Laplace equation is unique once we specify boundary conditions, and a formal proof of this will be given below. Suppose we have boundary conditions $\Phi(\mathbf{x}_S) = \Phi(\mathbf{x})|_S$ for any point $\mathbf{x}_S$ on S. Then if we require

$$\Phi_0(\mathbf{x}_S) = \Phi(\mathbf{x}_S) - \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{x}_S - \mathbf{x}'|}$$

there is a unique solution for Φ_0, and therefore a unique solution for Φ satisfying the given boundary conditions.

Alternatively, we may find the solution directly by solving

$$\nabla^2 \Phi = -\frac{q}{4\pi\epsilon_0} \delta^3(\mathbf{x} - \mathbf{x}')$$

with boundary conditions $\Phi_0(\mathbf{x}_S)$. This is more straightforward than it appears, because the Dirac delta function vanishes almost everywhere. Therefore, unless $\mathbf{x} = \mathbf{x}'$, we are solving the Laplace equation. As a result, we may construct our solution for Φ from solutions to the corresponding Laplace equation.

2.1 Example: Isolated point charge

The simplest example is for an isolated point charge, with the potential vanishing at infinity. We have already shown that

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi\delta^3(\mathbf{x} - \mathbf{x}')$$

so we immediately have

$$\Phi(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x})$$

where F satisfies the Laplace equation, $\nabla^2 F = 0$. By uniqueness, the function F must be determined by the boundary conditions. In the present case, we ask for $\Phi(\mathbf{x}, \mathbf{x}')$ to vanish at $\mathbf{x} \to \infty$. Since the first term in $\Phi(\mathbf{x}, \mathbf{x}')$ already satisfies this, we require the same condition for F:

$$\begin{aligned} \nabla^2 F &= 0 \\ F(\infty) &= 0 \end{aligned}$$

The argument of the preceeding section shows that $F(\mathbf{x}) = 0$ is the unique solution to this, so the Green function for an isolated point charge is $G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}$.

If we have a localized distribution of charge, $\rho(\mathbf{x}')$, in empty space, the potential vanishes at infinity and we can use this Green function to find the potential everywhere by integrating

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \int G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') d^3x' \\ &= \frac{1}{\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \end{aligned}$$

2.2 Example: Boundary conditions on a square

2.2.1 The Laplace equation

Consider a 2-dimensional example, with boundary conditions given on a square of side L with one corner at the origin. Let the boundary at $x = L$ have potential V_0, with the remaining boundary segments having

$\Phi = 0$. Then with the single charge q at $\mathbf{x'} = (x', y')$, the Poisson equation becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = -\frac{q}{4\pi\epsilon_0}\delta\left(x - x'\right)\delta\left(y - y'\right)$$

We begin by solving the homogeneous Laplace equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = 0$$

by separating in Cartesian coordinates. Writing $\Phi = X\left(x\right)Y\left(y\right)$, then dividing by Φ, the Laplace equation is

$$\begin{aligned} \frac{1}{XY}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)XY &= 0 \\ \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} &= 0 \end{aligned}$$

Since the first term depends only on x and the second only on y, each must be constant, so

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha^2$$

with the immediate solutions

$$\begin{aligned} X_\alpha\left(x\right) &= A_\alpha \sinh\alpha x + C_\alpha \cosh\alpha x \\ Y_\alpha\left(y\right) &= B_\alpha \sin\alpha y + D_\alpha \cos\alpha y \end{aligned}$$

These functions will satisfy the boundary conditions for y and at $x = 0$ if we set $\alpha = \frac{\pi n}{L}$ and $C_\alpha = D_\alpha = 0$, leaving

$$\begin{aligned} X_\alpha\left(x\right) &= A_\alpha \sinh\frac{n\pi x}{L} \\ Y_\alpha\left(y\right) &= B_\alpha \sin\frac{n\pi y}{L} \end{aligned}$$

Combining coefficients, the general solution is then a sum

$$\Phi\left(x, y\right) = \sum_n A_n \sinh\frac{n\pi x}{L}\sin\frac{n\pi y}{L}$$

The remaining boundary condition at $x = L$ is found by setting

$$V_0 = \Phi\left(L, y\right) = \sum_n A_n \sinh n\pi \sin\frac{n\pi y}{L}$$

This is just a Fourier series for a constant. Multiplying by $\sin\frac{m\pi y}{L}$ and integrating,

$$\begin{aligned} V_0\int_0^L dy\,\sin\frac{m\pi y}{L} &= \Phi\left(L, y\right) = \sum_n A_n \sinh n\pi\int_0^L dy\,\sin\frac{n\pi y}{L}\sin\frac{m\pi y}{L} \\ -\frac{LV_0}{m\pi}\left(\cos m\pi - 1\right) &= \sum_n A_n\frac{L}{2}\delta_{mn}\sinh n\pi \\ \frac{LV_0}{m\pi}\left(1 - \left(-1\right)^m\right) &= A_m\frac{L}{2}\sinh m\pi \end{aligned}$$

so that

$$A_m = \frac{2V_0}{m\pi\sinh m\pi}\left(1 - \left(-1\right)^m\right)$$

and finally

$$\Phi\left(x, y\right) = \sum_{n\ odd}\frac{4V_0}{n\pi\sinh n\pi}\sinh\frac{n\pi x}{L}\sin\frac{n\pi y}{L}$$

2.2.2 Fourier representation of the Dirac delta function

Noting that the y-dependence is described by a sine series, we make use of the fact that the Dirac delta functional may also be written as a Fourier series,

$$\delta\left(y-y'\right)=\sum_{n}A_{n}\sin\frac{n\pi y}{L}$$

To find the A_n, multiply by $\sin\frac{m\pi y}{L}$ and integrate to find the coefficients,

$$\begin{aligned}
\int_{0}^{L}dy\delta\left(y-y'\right)\sin\frac{m\pi y}{L} &= \int_{0}^{L}dy\sum_{n}A_{n}\sin\frac{n\pi y}{L}\sin\frac{m\pi y}{L}\\
\sin\frac{m\pi y'}{L} &= \sum_{n}A_{n}\delta_{mn}\int_{0}^{L}dy\,\sin^{2}\frac{n\pi y}{L}\\
&= \sum_{n}A_{n}\frac{1}{2}\delta_{mn}\int_{0}^{L}dy\left(\sin^{2}\frac{n\pi y}{L}+1-\cos^{2}\frac{n\pi y}{L}\right)\\
&= \frac{1}{2}\sum_{n}A_{n}\delta_{mn}\int_{0}^{L}dy\left(1-\cos\frac{2n\pi y}{L}\right)\\
&= \frac{1}{2}\sum_{n}A_{n}\delta_{mn}L\\
&= \frac{L}{2}A_{m}
\end{aligned}$$

so that $A_m=\frac{2}{L}\sin\frac{m\pi y'}{L}$ and we have

$$\delta\left(y-y'\right)=\frac{2}{L}\sum_{n}\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L}\qquad(1)$$

Integrating twice shows that

$$\frac{d^{2}}{dy^{2}}\left(-\frac{2}{L}\sum_{n}\frac{L^{2}}{n^{2}\pi^{2}}\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L}\right)=\delta\left(y-y'\right)$$

2.2.3 An ansatz for a particular solution

While there are systematic approaches to solving the point particle Poisson equation in various coordinate systems, we take a simpler approach here.

Suppose we guess that we can find a solution of the form

$$\Phi_{p}\left(x,y\right)=-\frac{2L}{\pi^{2}}\sum_{n}\frac{1}{n^{2}}f_{n}\left(x\right)\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L}\qquad(2)$$

and substitute eqs.(1) and (2) into the Poisson equation

$$\begin{aligned}
\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)\Phi &= -\frac{q}{4\pi\epsilon_{0}}\delta\left(x-x'\right)\delta\left(y-y'\right)\\
\sum_{n}\left(-\frac{L^{2}}{\pi^{2}}\frac{1}{n^{2}}\frac{d^{2}f_{n}\left(x\right)}{dx^{2}}+f_{n}\left(x\right)\right)\frac{2}{L}\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L} &= -\frac{q}{4\pi\epsilon_{0}}\delta\left(x-x'\right)\frac{2}{L}\sum_{n}\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L}
\end{aligned}$$

and therefore, equating like terms,

$$-\frac{L^2}{n^2\pi^2}\frac{d^2 f_n}{dx^2} + f_n = -\frac{q}{4\pi\epsilon_0}\delta\left(x - x'\right)$$

Now expand f_n and the second delta function,

$$\begin{aligned} f_n\left(x\right) &= \frac{2}{L}\sum_m B_m \sin\frac{m\pi x}{L} \\ \delta\left(x - x'\right) &= \frac{2}{L}\sum_m \sin\frac{m\pi x'}{L}\sin\frac{m\pi x}{L} \end{aligned}$$

and substitute,

$$\begin{aligned} \frac{L^2}{n^2\pi^2}\frac{2}{L}\sum_m B_m \frac{m^2\pi^2}{L^2}\sin\frac{m\pi x}{L} + \frac{2}{L}\sum_m B_m \sin\frac{m\pi x}{L} &= -\frac{q}{4\pi\epsilon_0}\frac{2}{L}\sum_m \sin\frac{m\pi x'}{L}\sin\frac{m\pi x}{L} \\ \left(1 + \frac{m^2}{n^2}\right)B_m &= -\frac{q}{4\pi\epsilon_0}\sin\frac{m\pi x'}{L} \\ B_m &= -\frac{q}{4\pi\epsilon_0}\frac{n^2}{n^2 + m^2}\sin\frac{m\pi x'}{L} \end{aligned}$$

Therefore,

$$\Phi_p\left(x, y\right) = \frac{q}{\pi^3\epsilon_0}\sum_{m,n}\frac{1}{n^2 + m^2}\sin\frac{m\pi x'}{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L}$$

Finally, check that

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi_p\left(x, y\right) &= -\frac{q}{\pi^3\epsilon_0}\sum_{m,n}\frac{1}{n^2 + m^2}\left(\frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{L^2}\right)\sin\frac{m\pi x'}{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L} \\ &= -\frac{q}{\pi^3\epsilon_0}\frac{\pi^2}{L^2}\sum_m \sin\frac{m\pi x'}{L}\sin\frac{m\pi x}{L}\sum_n \sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L} \\ &= -\frac{q}{4\pi\epsilon_0}\delta\left(x - x'\right)\delta\left(y - y'\right) \end{aligned}$$

2.2.4 Boundary conditions

The potential $\Phi_p\left(x, y\right)$ is a particular solution to our Poissoin equation, but it does not satisfy the boundary condition at $x = L$, instead vanishing at all four boundary lines. To get the complete solution, we need to add the homogeneous solution that satisfies the boundary conditions. The full solution is therefore

$$\Phi\left(x, y\right) = \frac{q}{\pi^3\epsilon_0}\sum_{m,n}\frac{1}{n^2 + m^2}\sin\frac{m\pi x'}{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi y'}{L}\sin\frac{n\pi y}{L} + \sum_{n\,odd}\frac{4V_0}{n\pi\sinh n\pi}\sinh\frac{n\pi x}{L}\sin\frac{n\pi y}{L}$$

3 Superposition

The general problem of electrostatics is to solve the Poission equation,

$$\nabla^2\Phi = -\frac{1}{\epsilon_0}\rho\left(\mathbf{x}\right)$$

with given charge density $\rho(\mathbf{x})$ and boundary conditions $\Phi(\mathbf{x})|_S$. Knowing the solution for a point charge at $\mathbf{x}'$ allows us to do this immediately by taking the superposition of infinitesimal charges $\rho(\mathbf{x}')\, d^3x'$ and summing (integrating) over our entire volume:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{x}')\, d^3x'}{|\mathbf{x} - \mathbf{x}'|} + \Phi_0(\mathbf{x})$$

where Φ_0 satisfies the Laplace equation. We choose Φ_0 so that Φ satisfies the boundary condition. We will examine details of this solution in the next Note.