Review of electrostatics and magenetostatics

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1 Electrostatics

1.1 Coulomb's law and the electric field

Starting from Coulomb's law for the force produced by a charge Q at the origin on a charge q at $\mathbf{x}$,

$$\mathbf{F}(\mathbf{x}) = \frac{qQ}{4\pi\epsilon_0 |\mathbf{x}|^2}\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ is a unit vector pointing from Q toward q. We may generalize this to let the source charge Q be at an arbitrary postion $\mathbf{x}'$ by writing the distance between the charges as $|\mathbf{x}-\mathbf{x}'|$ and the unit vector from $Q$to q as

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$$

Then Coulomb's law becomes

$$\mathbf{F}(\mathbf{x}) = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}_i|^2}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$$

Define the electric field as the force per unit charge at any given position,

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &\equiv \frac{\mathbf{F}(\mathbf{x})}{q} \\ &= \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\end{aligned}$$

We think of the electric field as existing at each point in space, so that *any* charge q placed at $\mathbf{x}$ experiences a force $q\mathbf{E}(\mathbf{x})$.

Since Coulomb's law is linear in the charges, the electric field for multiple charges is just the sum of the fields from each,

$$\mathbf{E}(\mathbf{x}) = \sum_{i=1}^{n}\frac{q_i}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3}$$

Knowing the electric field is equivalent to knowing Coulomb's law.

To formulate the equivalent of Coulomb's law for a continuous distribution of charge, we introduce the *charge density*, $\rho(\mathbf{x})$. We can define this as the total charge per unit volume for a volume centered at the position $\mathbf{x}$, in the limit as the volume becomes "small". What we mean by small is any size much smaller than the size over which ρ is changing, but large enough that the volume still contains many charges.

$$\rho(\mathbf{x}) = \lim_{\Delta V\rightarrow small}\frac{Q\,in\,\Delta V\,about\,\mathbf{x}}{\Delta V}$$

Then, in a volume ΔV, the total charge is $\rho(\mathbf{x})\,\Delta V$. The continuous idealization is good enough that we may write this infinitesimally,

$$dQ = \rho(\mathbf{x})\, d^3x$$

With this preparation, we reconsider the electric field, replacing q_i with ρd^3x. Let q_i be replaced by $q_i = q(\mathbf{x}_i) = \rho(\mathbf{x}_i)\,\Delta V$ and take the infinitesimal limit as $\Delta V \to 0$ and $n \to \infty$,

$$\mathbf{E}(\mathbf{x}) = \lim_{\Delta V \to 0} \sum_{i=1}^{n} \frac{\rho(\mathbf{x}_i)\,\Delta V_i}{4\pi\epsilon_0} \frac{\mathbf{x}-\mathbf{x}_i}{|\mathbf{x}-\mathbf{x}_i|^3}$$

In this limit, the sum becomes an integral, as the charge positions $\mathbf{x}_i \to \mathbf{x}'$ vary smoothly over all space,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

1.2 The electric flux

It is useful to compute the electric flux over an arbitrary closed surface, S,

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, d^2x$$

where $\hat{\mathbf{n}}$ is the unit outward normal to S. This is easiest to find for a single charge, then use superposition to get the general expression. For a point charge at the origin,

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, d^2x = \oint_S \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\, d^2x$$

Approximate the surface S by small pieces with normals either parallel or perpendicular to the radial unit vector $\hat{\mathbf{r}}$. Sections orthogonal to $\hat{\mathbf{r}}$ give no conntribution, while the pieces of spherical surfaces with normals parallel to $\hat{\mathbf{r}}$ have

$$\begin{aligned} \hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\, d^2x &= \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\, r^2 d\Omega \\ &= r^2 \sin\theta d\theta d\varphi \end{aligned}$$

where $d\Omega = \sin\theta d\theta d\varphi$ is an elemental solid angle on a sphere. The flux then becomes

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, d^2x = \frac{Q}{4\pi\epsilon_0}\oint_S \sin\theta d\theta d\varphi$$

The integral is now just the area, 4π, of a unit sphere and we have Gauss's law for a single charge,

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, d^2x = \frac{Q}{\epsilon_0}$$

Since the surface is arbitrary and the law is linear, we may add together the contributions of many charges to get

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\, d^2x = \frac{Q_{enclosed}}{\epsilon_0}$$

where $Q_{enclosed}$ is the total charge inside S. This is *Gauss's law.*

In the continuum limit, the enclosed charge satisfies

$$Q_{enclosed} = \int \rho(\mathbf{x})\, d^3x$$

1.3 Line integral of the electric field

Again consider the electric field of a point charge at the origin,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$$

Consider the integral of $\mathbf{E} \cdot d\mathbf{l}$ along an arbitrary curve, C,

$$\begin{aligned}
\int\limits_C \mathbf{E} \cdot d\mathbf{l} &= \frac{Q}{4\pi\epsilon_0}\int\limits_C \frac{1}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{l} \\
&= \frac{Q}{4\pi\epsilon_0}\int\limits_{r_i}^{r_f} \frac{1}{r^2}dr \\
&= \frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{r_f} + \frac{1}{r_i}\right)
\end{aligned}$$

where r_i and r_f are the initial and final radii of the curve.

Now suppose the curve is a closed loop. Then $r_i = r_f$ and the integral vanishes, regardless of the closed curve, C,

$$\oint\limits_C \mathbf{E} \cdot d\mathbf{l} = \frac{Q}{4\pi\epsilon_0}\oint\limits_C \frac{1}{r^2}dr = 0$$

Notice that this result depends only on the relative position of the curve and the charge, not on the charge being at the origin.

Now, suppose there is more than one charge. Since the electric field $\mathbf{E}$ is the sum of the fields from each charge, $\mathbf{E}_i$, and the line integral for each vanishes, the sum vanishes,

$$\oint\limits_C \mathbf{E} \cdot d\mathbf{l} = \sum_{i=1}^{n}\oint\limits_C \mathbf{E}_i \cdot d\mathbf{l} = 0$$

as long as the curve doesn't pass through any of the charges. The result holds equally well in the limit of a charge density, and we conclude that

$$\oint\limits_C \mathbf{E} \cdot d\mathbf{l} = 0$$

for any curve C in empty space.

1.4 The integral and differential forms of the laws

We now have the integral form of the Maxwell equations for electrostatics:

$$\begin{aligned}
\oint\limits_S \mathbf{E} \cdot \hat{\mathbf{n}}\, d^2x &= \frac{Q_{enclosed}}{\epsilon_0} \\
\oint\limits_C \mathbf{E} \cdot d\mathbf{l} &= 0
\end{aligned}$$

These may be used to find the electric field when the charge distribution is sufficiently symmetrical, but are of little help for general charge distributions. A much more powerful formulation is to recast these equations as equivalent differential equations.

1.4.1 The divergence of the electric field

Starting from the integral form of Gauss's law, we treat the charge as a continuous distribution, $\rho(\mathbf{x})$. Then, letting V be the volume enclosed by the arbitrary closed surface S, and substituting the integral for $Q_{enclosed}$,

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, d^2x = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x})\, d^3x$$

Applying the divergence theorem to the left side this becomes

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E}\, d^3x = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x})\, d^3x$$

Combining the integrals, we have

$$\int_V \left[\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0}\right] d^3x = 0$$

where V is an *arbitrary* volume.

We now prove by contradiction that the integrand must be zero everywhere in V. Suppose there is some point $\mathcal{P}$ in V where $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0} > 0$. Then, since we expect $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0}$ to be continuous, there must be a region around $\mathcal{P}$ over which $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0}$ remains positive. Take the arbitrary volume V to be within this region. Then the integral is necessarily positive, and we have a contradition. A similar argument holds if $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0} < 0$, so we must have $\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{\rho(\mathbf{x})}{\epsilon_0} = 0$ at $\mathcal{P}$. Since this argument holds for any point in the region, the integrand must vanish everywhere, and we have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho(\mathbf{x})$$

This is the differential form of Gauss's law. It will be extremely useful once we also know about the curl of $\mathbf{E}$.

1.4.2 The curl of the electric field

Now apply Stokes' theorem to the line integral. We have

$$\begin{aligned} 0 &= \oint_C \mathbf{E} \cdot d\mathbf{l} \\ &= \iint_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{n}\, d^2x \end{aligned}$$

where S is now an arbitrary surface with boundary C. An argument similar to the one above shows that this can only be the case for all surfaces if the integrand vanishes. Moreover, since $\mathbf{n}$ is arbitrary as well, we have

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

in free space for the electric field of any static charge distribution.

1.5 The electric potential

The vanishing of closed line integrals of the electric field, $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$, (or equivalently, the vanishing curl of $\mathbf{E}$, since Stokes' theorem makes these statements equivalent), means that we may define a function from the integral of the electric field along curves,

$$\Phi(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E} \cdot d\mathbf{l}$$

This integral is independent of our choice of the path of integration, and therefore depends only on the endpoint $\mathbf{x}$. To see this, subtract the integral along any two curves between the same endpoints,

$$\Phi_{C_1}(\mathbf{x}) - \Phi_{C_2}(\mathbf{x}) = -\int\limits_{\mathbf{x}_0, C_1}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l} + \int\limits_{\mathbf{x}_0, C_2}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l}$$

Since integrating along C_1 from $\mathbf{x}_0$ to $\mathbf{x}$ just gives the negative of the integral along $-C_1$ from $\mathbf{x}$ to $\mathbf{x}_0$, we may combine the two integrals on the right into a single closed line integral, which then vanishes:

$$-\int\limits_{\mathbf{x}_0, C_1}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l} + \int\limits_{\mathbf{x}_0, C_2}^{\mathbf{x}} \mathbf{E}\cdot d\mathbf{l} = \oint\limits_{C_2 - C_1} \mathbf{E}\cdot d\mathbf{l} = 0$$

Therefore, $\Phi_{C_1} = \Phi_{C_2}$ for any two paths between the same endpoints, and $\Phi(\mathbf{x})$ is a function – its value depends only on $\mathbf{x}$ and not on the curve of integration.

$V(\mathbf{x})$ is called the *electric potential.* From it, we may find the electric field by taking the gradient,

$$\mathbf{E} = -\boldsymbol{\nabla} V(\mathbf{x}) \tag{1}$$

1.5.1 An alternative proof

An alternative proof, which gives an explicit formula for the potential, follows from the gradient of $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$,

$$\begin{aligned}
\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
&= -\frac{1}{2}\frac{1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right] \\
&= -\frac{1}{2}\frac{1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}\left[2(x-x')\hat{\mathbf{i}} + 2(y-y')\hat{\mathbf{j}} + 2(z-z')\hat{\mathbf{k}}\right] \\
&= -\frac{1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}\left[(x-x')\hat{\mathbf{i}} + (y-y')\hat{\mathbf{j}} + (z-z')\hat{\mathbf{k}}\right] \\
&= -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}
\end{aligned}$$

Using this, we may immediately write the electric field of a charge distribution as

$$\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}d^3x' \\
&= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\left(-\boldsymbol{\nabla}\frac{1_i}{|\mathbf{x}-\mathbf{x}_i|}\right)d^3x'
\end{aligned}$$

Since the gradient involves derivatives with respect to $\mathbf{x}$ while the integral is over $\mathbf{x}'$, we may interchange the integral and gradient. Then

$$\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}\Phi$$

where

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\limits_V \frac{\rho(\mathbf{x}')\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

This scalar integration is generally easier than the vector integration for finding the electric field directly. Once we have the potential, we easily find the electric field by differentiating.

2 Maxwell's equations for electrostatics

While the curl of $\mathbf{E}$ maybe be different from zero in the presence of a changing magnetic field, Maxwell's equations for *electrostatics* reduce to

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \rho(\mathbf{x}) \\ \boldsymbol{\nabla} \times \mathbf{E} &= 0 \end{aligned} \tag{2}$$

The Helmholz theorem tells us that knowing the divergence and curl of a vector field, together with boundary conditions, uniquely determines the field everywhere within the boundary. These equations therefore give a complete characterization of the electric field once we specify the charge density $\rho(\mathbf{x})$ in a volume V, and give boundary conditions on the boundary of V.

As we have seen above, the vanishing curl of $\mathbf{E}$ implies the existence of a potential. Furthermore, we may write the electrostatic equations in terms of the potential, eq.(1). Substituting this into the electrostatic equations, the curl of the gradient vanishes automatically, while Gauss's law becomes

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho(\mathbf{x}) \tag{3}$$

This is the Poisson equation. Together with boundary conditions, this is gives a unique solution for the potential, which then determines the electric field. We will devote considerable attention to solving the Poisson equation.The electric field is then found from $\mathbf{E} = -\boldsymbol{\nabla}\Phi(\mathbf{x})$.

3 Magnetostatics

The force on charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

It is also moving charges which *produce* magnetic fields. To quantify this, we first need to characterize moving charge distributions. It is most useful to treat the current as a *current density vector* $\mathbf{J}$ rather than a simple current scalar, I, since the current may vary in both direction and magnitude from place to place. The current density captures both of these features.

3.1 Current density

A current I may be viewed as made up of many charges in a (microscopically large, macroscopically small) region d^3x moving with velocity $\mathbf{v}(\mathbf{x})$. If the density of charges at $\mathbf{x}$ is $\rho(\mathbf{x})$, then there is a current density, $\mathbf{J} = \rho\mathbf{v}$. Let d^2x be an arbitrary surface element with unit normal $\hat{\mathbf{n}}$. Then the component of the current density crossing this surface is $\mathbf{J} \cdot \hat{\mathbf{n}}$; this is the amount of charge per unit time per unit area crossing d^2x, so the charge per unit time crossing the surface is the product with the area,

$$dI = \left.\frac{dq}{dt}\right|_{through\, d^2x} = \mathbf{J} \cdot \hat{\mathbf{n}} d^2x$$

Integrating over an arbitrary surface S, we get the total current

$$I = \iint\limits_S \mathbf{J} \cdot \hat{\mathbf{n}} d^2x$$

Now suppose we have a region of space with charge density ρ. Let some or all of this charge move as a current density, $\mathbf{J}$. Now, since we find that total charge is conserved, we know that the total charge in some volume $\mathcal{V}$ can only change if the current carries charge across the boundary $\mathcal{S}$ if $\mathcal{V}$. Therefore, with the charge in the volume $\mathcal{V}$ given by

$$Q_{tot} = \int_{\mathcal{V}} \rho d^3x$$

the time rate of change of Q_{tot} must be given by the total flux $\mathbf{J}$ across the boundary. Let $\hat{\mathbf{n}}$ be the outward normal of the boundary $\mathcal{S}$ of $\mathcal{V}$. Then

$$\frac{dQ_{tot}}{dt} = -\oint_{\mathcal{S}} \mathbf{J}\cdot\hat{\mathbf{n}}d^2x \tag{4}$$

This expresses conservation of charge.

On the left side of the conservation law, rewrite $\frac{dQ_{tot}}{dt}$ by interchanging the order of integration and differentiation ,

$$\begin{aligned}\frac{dQ_{tot}}{dt} &= \frac{d}{dt}\int_{\mathcal{V}} \rho d^3x \\ &= \int_{\mathcal{V}} \frac{\partial\rho}{\partial t} d^3x\end{aligned}$$

while on the right we use the divergence theorem, $-\oint_{\mathcal{S}} \mathbf{J}\cdot\hat{\mathbf{n}}d^2x = -\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{J}d^3x$. Substituting both these changes into eq.(4), we have

$$\int_{\mathcal{V}} \left(\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}\right) d^3x = 0$$

Since the final equation holds for all volumes $\mathcal{V}$ it must hold at each point, leading us to the *continuity equation*:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0 \tag{5}$$

An equation of this sort holds anytime there is a conserved quantity.

We define a *steady state current* to be one for which ρ and $\mathbf{J}$ are independent of explicit time dependence, $\frac{\partial\rho}{\partial t} = 0, \frac{\partial\mathbf{J}}{\partial t} = 0$. For a steady state current, the current density has vanishing divergence, $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$.

3.2 The Biot-Savart law and magnetostatics

Careful measurements of the forces on moving charges shows that a charge density $\mathbf{J}$ produces a magnetic field according to the Biot-Savart law.Next, we consider the effect of a current in *producing* a magnetic field. The experimental results are summarized for steady state line currents by the *Biot-Savart law*,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

The constant μ_0 is the *permeability of free space* with the value $\mu_0 = 4\pi\times 10^{-7}N/A^2$. There is a clear parallel with our equation for the electric field. Instead of simply the charge density times the factor $\frac{\mathbf{x}_0-\mathbf{x}}{|\mathbf{x}_0-\mathbf{x}|^3}$, we now require the cross product with the current density.

In the Notes on Ampere's law is a calculation of the curl of $\mathbf{B}(\mathbf{x})$ directly from the Biot-Savart law. The result is the differential form of Ampère's law for steady currents,

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$$

If we compute the flux of this equation across any surface S, the right side gives μ_0 times the total current passing through the surface. Therefore,

$$\begin{aligned} \mu_0 I_{enclosed} &= \mu_0 \int \mathbf{J} \cdot \hat{\mathbf{n}} d^2 x \\ &= \int (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \hat{\mathbf{n}} d^2 x \end{aligned}$$

Applying Stokes' theorem gives the integral of $\mathbf{B}$ around the boundary C of S,

$$\mu_0 I_{enclosed} = \oint \mathbf{B} \cdot d\mathbf{l}$$

This is the integral form of Ampère's law for magnetostatics.

The converse holds as well. Given the integral form, we may reverse the steps to derive the integral form.

3.3 The divergence of the magnetic field

We now find the divergence of $\mathbf{B}$. Start again with the general form of the Biot-Savart law,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3 x'$$

and take the divergence of both sides with respect to $\mathbf{x}$,

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{x}} \cdot \mathbf{B}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla}_{\mathbf{x}} \cdot \left[\frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \right] d^3 x' \\ &= -\frac{\mu_0}{4\pi} \int \boldsymbol{\nabla}_{\mathbf{x}} \cdot \left[\mathbf{J}(\mathbf{x}') \times \boldsymbol{\nabla}_{\mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right] d^3 x' \end{aligned}$$

Now we need to rearrange terms. The divergence of a cross product may be rewritten as

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$$

so we may rewrite the integrand as

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{x}} \cdot \left[\mathbf{J}(\mathbf{x}') \times \boldsymbol{\nabla}_{\mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right] &= -\boldsymbol{\nabla}_{\mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \left[\boldsymbol{\nabla}_{\mathbf{x}} \times \mathbf{J}(\mathbf{x}') \right] + \mathbf{J}(\mathbf{x}') \cdot \left[\boldsymbol{\nabla}_{\mathbf{x}} \times \boldsymbol{\nabla}_{\mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right] \\ &= 0 \end{aligned}$$

where the first term on the right containing $\boldsymbol{\nabla}_{\mathbf{x}} \times \mathbf{J}(\mathbf{x}')$ vanishes immediately because $\mathbf{J}(\mathbf{x}')$ does not depend on the observation point $\mathbf{x}$. The second term also vanishes because the curl of a gradient is zero. Therefore, the divergence of the magnetic field vanishes:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

It turns out that this law applies even in non-steady state situations.

Integrating the divergence of $\mathbf{B}$ and using the divergence theorem

$$\begin{aligned} 0 &= \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{B} d^3 x \\ &= \oint_{S} \mathbf{B} \cdot \mathbf{n} d^2 x \end{aligned}$$

which shows that there is no net magnetic flux across any closed surface. In particular, there are no isolated magnetic charges (monopoles).

4 The equations of magnetostatics

Summarizing the magnetostatic equations, we have

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{J}(\mathbf{x})\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0\end{aligned}$$

together with the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

These equations, together with boundary conditions, uniquely determine static magnetic fields.

The integral forms of the magnetostatic laws are therefore,

$$\begin{aligned}\oint\limits_{\mathcal{C}}\mathbf{B}\cdot d\mathbf{l} &= \mu_0 I_{\mathcal{S}}\\ \oint\limits_{S}\mathbf{B}\cdot\mathbf{n}d^2x &= 0\end{aligned}$$

These integral forms are useful for cases of high symmetry and for establishing the boundary conditions.

4.1 The vector potential for the magnetic field

As we found for the electric field, it simplifies calculations to write the magnetic field in terms of a potential. However, while the electric field has vanishing curl and a source for its divergence, the magnetic field has the opposite: a source for the curl and a vanishing divergence. Therefore, we make use of the vanishing divergence of $\mathbf{B}$ to write $\mathbf{B}$ as a curl.

Clearly, if we set

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

for some vector field $\mathbf{A}$, the divergence will vanish automatically because the divergence of a curl is identically zero. Conversely, the Helmholz theorem shows that if the divergence vanishes then $\mathbf{B}$ may be written as a curl of some vector. The vector field $\mathbf{A}$ is called the *vector potential.*

Writing $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ automatically ensures that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. We now substitute this into Ampère's law and use the identity for a double curl:

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \mu_0\mathbf{J}(\mathbf{r})\\ \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} &= \mu_0\mathbf{J}(\mathbf{r})\end{aligned}$$

We can eliminate the first term by using *gauge freedom.*

Gauge freedom arises because there is more than one allowed vector potential. If $\mathbf{A}$ is any vector field satisfying $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, and we set $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}f$ for any function f, then it is also true that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}'$. The choice of the function f is called the gauge, and this choice has no effect on the measurable magnetic field, must like our freedom to add a constant to a scalar potential.

To use the gauge freedom to simplify the form of Ampère's law, first suppose we have any vector potential $\mathbf{A}_0(\mathbf{x})$ satisfying $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}_0$. Furthermore, it will be the case that the divergence of $\mathbf{A}_0$ is some function $g(\mathbf{x})$,

$$\boldsymbol{\nabla}\cdot\mathbf{A}_0 = g(\mathbf{x})$$

Let $\mathbf{A}$ be another allowed vector potential related to $\mathbf{A}_0$ by

$$\mathbf{A} = \mathbf{A}_0 + \boldsymbol{\nabla}f$$

where f is a function of our choosing. We would like to choose the function f so that the divergence of the new potential vanishes. This requires

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot \mathbf{A} \\ &= \boldsymbol{\nabla} \cdot (\mathbf{A}_0 + \boldsymbol{\nabla} f) \\ &= g + \nabla^2 f \end{aligned}$$

so that f must satisfy the Poisson equation,

$$\nabla^2 f = -g$$

where g is the known divergence of our original vector potential $\mathbf{A}_0$. We have techniques for solving the Poisson equation, so we can always find the required function f.

We now have a vector potential satisfying two conditions:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{A} &= \mathbf{B} \\ \boldsymbol{\nabla} \cdot \mathbf{A} &= 0 \end{aligned}$$

Substituting into Ampère's law now gives the simpler result,

$$\begin{aligned} \mu_0 \mathbf{J}(\mathbf{r}) &= \boldsymbol{\nabla} \times \mathbf{B} \\ &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ &= -\nabla^2 \mathbf{A} \end{aligned}$$

and vanish, and Ampère's law is

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}(\mathbf{r})$$

This is just three copies of the Poisson equation, one for each component, so in principal we know how to solve the equations of magnetostatics.

For fields vanishing at infinity, the solution is

$$\mathbf{A}(\mathbf{x}_0) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x})}{|\mathbf{x}_0 - \mathbf{x}|} d^3 x$$

For other cases, we know that the solutions are unique, once we specify boundary conditions.

5 The Poisson equation

The preceding sections have shown that the electric field may be found by solving the Poisson equation,

$$\nabla^2 \Phi = -\frac{1}{\epsilon_0} \rho(\mathbf{x})$$

for specified boundary conditions, then taking the gradient to find the field,

$$\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla} \Phi$$

To find the magnetic field, we solve the vector Poisson equation,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}(\mathbf{x})$$

then take the curl to find the magnetic field,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

Therefore, magnetostatics and electrostatics both lead us to consider solutions to the Poisson equations, including a careful treatment of boundary conditions.