The Levi-Civita tensor

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In 3-dimensions, we define the Levi-Civita tensor, ε_{ijk}, to be totally antisymmetric, so we get a minus sign under interchange of any pair of indices. We work throughout in Cartesian coordinate. This means that most of the 27 components are zero, since, for example,

$$\varepsilon_{212} = -\varepsilon_{212}$$

if we imagine interchanging the two 2s. This means that the only nonzero components are the ones for which i, j and k all take different value. There are only six of these, and all of their values are determined once we choose any one of them. Define

$$\varepsilon_{123} \equiv 1$$

Then by antisymmetry it follows that

$$\begin{aligned} \varepsilon_{123} &= \varepsilon_{231} = \varepsilon_{312} = +1 \\ \varepsilon_{132} &= \varepsilon_{213} = \varepsilon_{321} = -1 \end{aligned}$$

All other components are zero.

Using ε_{ijk} we can write index expressions for the cross product and curl. The i^{th} component of the cross product is given by

$$[\mathbf{u} \times \mathbf{v}]_i = \varepsilon_{ijk} u_j v_k$$

as we check by simply writing out the sums for each value of i,

$$\begin{aligned} [\mathbf{u} \times \mathbf{v}]_1 &= \varepsilon_{1jk} u_j v_k \\ &= \varepsilon_{123} u_2 v_3 + \varepsilon_{132} u_3 v_2 + (\textit{all other terms are zero}) \\ &= u_2 v_3 - u_3 v_2 \\ [\mathbf{u} \times \mathbf{v}]_2 &= \varepsilon_{2jk} u_j v_k \\ &= \varepsilon_{231} u_3 v_1 + \varepsilon_{213} u_1 v_3 \\ &= u_3 v_1 - u_1 v_3 \\ [\mathbf{u} \times \mathbf{v}]_3 &= \varepsilon_{3jk} u_j v_k \\ &= u_1 v_2 - u_2 v_1 \end{aligned}$$

We get the curl simply by replacing u_i by $\nabla_i = \frac{\partial}{\partial x_i}$,

$$[\nabla \times \mathbf{v}]_i = \varepsilon_{ijk} \nabla_j v_k$$

If we sum these expressions with basis vectors $\mathbf{e}_i$, where $\mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}, \mathbf{e}_3 = \mathbf{k}$, we may write these as vectors:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= [\mathbf{u} \times \mathbf{v}]_i \mathbf{e}_i \\ &= \varepsilon_{ijk} u_j v_k \mathbf{e}_i \\ \nabla \times \mathbf{v} &= \varepsilon_{ijk} \mathbf{e}_i \nabla_j v_k \end{aligned}$$

There are useful identities involving pairs of Levi-Civita tensors. The most general is

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}$$

To check this, first notice that the right side is antisymmetric in i, j, k and antisymmetric in l, m, n. For example, if we interchange i and j, we get

$$\varepsilon_{jik}\varepsilon_{lmn} = \delta_{jl}\delta_{im}\delta_{kn} + \delta_{jm}\delta_{in}\delta_{kl} + \delta_{jn}\delta_{il}\delta_{km} - \delta_{jl}\delta_{in}\delta_{km} - \delta_{jn}\delta_{im}\delta_{kl} - \delta_{jm}\delta_{il}\delta_{kn}$$

Now interchange the first pair of Kronecker deltas in each term, to get i, j, k in the original order, then rearrange terms, then pull out an overall sign,

$$\begin{aligned}
\varepsilon_{jik}\varepsilon_{lmn} &= \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jm}\delta_{kl} + \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jn}\delta_{kl} - \delta_{il}\delta_{jm}\delta_{kn} \\
&= -\delta_{il}\delta_{jm}\delta_{kn} - \delta_{im}\delta_{jn}\delta_{kl} - \delta_{in}\delta_{jl}\delta_{km} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl} + \delta_{im}\delta_{jl}\delta_{kn} \\
&= -\left(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}\right) \\
&= -\varepsilon_{ijk}\varepsilon_{lmn}
\end{aligned}$$

Total antisymmetry means that if we know one component, the others are all determined uniquely. Therefore, set $i = l = 1, j = m = 2, k = n = 3$, to see that

$$\begin{aligned}
\varepsilon_{123}\varepsilon_{123} &= \delta_{11}\delta_{22}\delta_{33} + \delta_{12}\delta_{23}\delta_{31} + \delta_{13}\delta_{21}\delta_{32} - \delta_{11}\delta_{23}\delta_{32} - \delta_{13}\delta_{22}\delta_{31} - \delta_{12}\delta_{21}\delta_{33} \\
&= \delta_{11}\delta_{22}\delta_{33} \\
&= 1
\end{aligned}$$

Check one more case. Let $i = 1, j = 2, k = 3$ again, but take $l = 3, m = 2, n = 1$. Then we have

$$\begin{aligned}
\varepsilon_{123}\varepsilon_{321} &= \delta_{13}\delta_{22}\delta_{31} + \delta_{12}\delta_{21}\delta_{33} + \delta_{11}\delta_{23}\delta_{32} - \delta_{13}\delta_{21}\delta_{32} - \delta_{11}\delta_{22}\delta_{33} - \delta_{12}\delta_{23}\delta_{31} \\
&= -\delta_{11}\delta_{22}\delta_{33} \\
&= -1
\end{aligned}$$

as expected.

We get a second identity by setting $n = k$ and summing,

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmk} &= \delta_{il}\delta_{jm}\delta_{kk} + \delta_{im}\delta_{jk}\delta_{kl} + \delta_{ik}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jk}\delta_{km} - \delta_{ik}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kk} \\
&= 3\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} - \delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} \\
&= (3 - 1 - 1)\,\delta_{il}\delta_{jm} - (3 - 1 - 1)\,\delta_{im}\delta_{jl} \\
&= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}
\end{aligned}$$

so we have a much simpler, and very useful, relation

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

A second sum gives another identity. Setting $m = j$ and summing again,

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{ljk} &= \delta_{il}\delta_{mm} - \delta_{im}\delta_{ml} \\
&= 3\delta_{il} - \delta_{il} \\
&= 2\delta_{il}
\end{aligned}$$

Setting the last two indices equal and summing provides a check on our normalization,

$$\varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{ii} = 6$$

This is correct, since there are only six nonzero components and we are summing their squares.

Collecting these results,

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} \\
\varepsilon_{ijk}\varepsilon_{lmk} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\
\varepsilon_{ijk}\varepsilon_{ljk} &= 2\delta_{il} \\
\varepsilon_{ijk}\varepsilon_{ijk} &= 6
\end{aligned}
$$

Now we use these properties to prove some vector identities. First, consider the triple product,

$$
\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= u_i\left[\mathbf{v}\times\mathbf{w}\right]_i \\
&= u_i\varepsilon_{ijk}v_jw_k \\
&= \varepsilon_{ijk}u_iv_jw_k
\end{aligned}
$$

Because $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$, we may write this in two other ways,

$$
\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \varepsilon_{ijk}u_iv_jw_k \\
&= \varepsilon_{kij}u_iv_jw_k \\
&= w_k\varepsilon_{kij}u_iv_j \\
&= w_i\left[\mathbf{u}\times\mathbf{v}\right]_i \\
&= \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \varepsilon_{ijk}u_iv_jw_k \\
&= \varepsilon_{jki}u_iv_jw_k \\
&= v_j\left[\mathbf{w}\times\mathbf{u}\right]_j \\
&= \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})
\end{aligned}
$$

so that we have established

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$$

and we get the negative permutations by interchanging the order of the vectors in the cross products.

Next, consider a double cross product:

$$
\begin{aligned}
\left[\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\right]_i &= \varepsilon_{ijk}u_j\left[\mathbf{v}\times\mathbf{w}\right]_k \\
&= \varepsilon_{ijk}u_j\varepsilon_{klm}v_lw_m \\
&= \varepsilon_{ijk}\varepsilon_{klm}u_jv_lw_m \\
&= \varepsilon_{ijk}\varepsilon_{lmk}u_jv_lw_m \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)u_jv_lw_m \\
&= \delta_{il}\delta_{jm}u_jv_lw_m - \delta_{im}\delta_{jl}u_jv_lw_m \\
&= \left(\delta_{il}v_l\right)\left(\delta_{jm}u_jw_m\right) - \left(\delta_{jl}u_jv_l\right)\left(\delta_{im}w_m\right) \\
&= v_i\left(u_mw_m\right) - \left(u_jv_j\right)w_i
\end{aligned}
$$

Returning to vector notation, this is the $BAC - CAB$ rule,

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$$

Finally, look at the curl of a cross product,

$$\left[\nabla\times(\mathbf{v}\times\mathbf{w})\right]_i = \varepsilon_{ijk}\nabla_j\left[\mathbf{v}\times\mathbf{w}\right]_k$$

$$
\begin{aligned}
&= \varepsilon_{ijk}\nabla_j\left(\varepsilon_{klm}v_l w_m\right) \\
&= \varepsilon_{ijk}\varepsilon_{klm}\left(\left(\nabla_j v_l\right) w_m + v_l \nabla_j w_m\right) \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\left(\left(\nabla_j v_l\right) w_m + v_l \nabla_j w_m\right) \\
&= \delta_{il}\delta_{jm}\left(\left(\nabla_j v_l\right) w_m + v_l \nabla_j w_m\right) - \delta_{im}\delta_{jl}\left(\left(\nabla_j v_l\right) w_m + v_l \nabla_j w_m\right) \\
&= \left(\nabla_m v_i\right) w_m + v_i \nabla_m w_m - \left(\nabla_j v_j\right) w_i - v_j \nabla_j w_i
\end{aligned}
$$

Restoring the vector notation, we have

$$\nabla \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \nabla)\,\mathbf{v} + (\nabla \cdot \mathbf{w})\,\mathbf{v} - (\nabla \cdot \mathbf{v})\,\mathbf{w} - (\mathbf{v} \cdot \nabla)\,\mathbf{w}$$

If you doubt the advantages here, try to prove these identities by explicitly writing out all of the components!