Yukawa potential

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1 The Yukawa potential

We consider properties of the Yukawa potential,

$$V(r) = \frac{k}{r} e^{-\frac{r}{a}}$$

This potential is the static, spherically symmetric solution to the Klein-Gordon equation,

$$-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + \nabla^2 V = \frac{m^2 c^2}{\hbar^2} V$$

To see this, let $V = V(r)$ and write the Laplacian in spherical coordinates. Then we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = \frac{m^2 c^2}{\hbar^2} V$$

Differentiating on the left side, we have

$$\frac{d^2 V}{dr^2} + \frac{2}{r}\frac{dV}{dr} - \frac{m^2 c^2}{\hbar^2} V = 0$$

Let $U = rV$. Then

$$\begin{aligned} \frac{d^2 U}{dr^2} &= \frac{d}{dr}\left(V + r\frac{dV}{dr}\right) \\ &= 2\frac{dV}{dr} + r\frac{d^2 V}{dr^2} \end{aligned}$$

so that we may write the equation as

$$\begin{aligned} \frac{1}{r}\frac{d^2 U}{dr^2} - \frac{m^2 c^2}{\hbar^2}\frac{U}{r} &= 0 \\ \frac{d^2 U}{dr^2} - \frac{m^2 c^2}{\hbar^2} U &= 0 \end{aligned}$$

and this has exponential solutions,

$$U = U_0 \exp\left(\pm \frac{mc}{\hbar} r\right)$$

Choosing the decaying exponential for our solution, we have

$$V = -\frac{k}{r} e^{-\frac{r}{a}}$$

where $a = \frac{\hbar}{mc}$, the reduced Compton wavelength of a particle of mass m.

2 Bound orbits

We know from our general results that the conserved energy and angular momentum are given by

$$\begin{aligned} E &= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r}e^{-\frac{r}{a}} \\ L &= \mu r^2\dot{\varphi} \end{aligned}$$

and from the energy expression we see that the radial motion is described by the effective potential

$$V_{eff} = \frac{L^2}{2\mu r^2} - \frac{k}{r}e^{-\frac{r}{a}}$$

Bound orbits exist if there is a minimum of the effective potential:

$$\begin{aligned} V(r) &= \frac{L^2}{2\mu r^2} - \frac{k}{r}e^{-\frac{r}{a}} \\ 0 &= \frac{dV}{dr} \\ &= -\frac{L^2}{\mu r^3} + \frac{k}{r^2}e^{-\frac{r}{a}} + \frac{k}{ar}e^{-\frac{r}{a}} \\ &= -\frac{L^2}{\mu r^3} + \left(\frac{k}{r^2} + \frac{k}{ar}\right)e^{-\frac{r}{a}} \end{aligned}$$

This has solutions iff

$$\frac{L^2}{\mu} = kr\left(1 + \frac{r}{a}\right)e^{-\frac{r}{a}}$$

Since the right side is positive definite, there is always a value of L small enough that the equation is satisfied. Computing the second derivative at this point, we have

$$\begin{aligned} V' &= -\frac{L^2}{\mu r^3} + \left(\frac{k}{r^2} + \frac{k}{ar}\right)e^{-\frac{r}{a}} \\ \frac{d^2V}{dr^2} &= \frac{3L^2}{\mu r^4} + \left(-\frac{2k}{r^3} - \frac{k}{ar^2}\right)e^{-\frac{r}{a}} - \frac{1}{a}\left(\frac{k}{r^2} + \frac{k}{ar}\right)e^{-\frac{r}{a}} \\ &= \frac{3L^2}{\mu r^4} - \frac{k}{r}\left(\frac{2}{r^2} + \frac{2}{ar} + \frac{1}{a^2}\right)e^{-\frac{r}{a}} \\ &= \frac{3}{r^4}kr\left(1 + \frac{r}{a}\right)e^{-\frac{r}{a}} - \frac{k}{r}\left(\frac{2}{r^2} + \frac{2}{ar} + \frac{1}{a^2}\right)e^{-\frac{r}{a}} \\ &= \frac{k}{r}\left(\frac{3}{r^2} + \frac{3}{ar} - \left(\frac{2}{r^2} + \frac{2}{ar} + \frac{1}{a^2}\right)\right)e^{-\frac{r}{a}} \\ &= \frac{k}{r^3}\left(1 + \frac{r}{a} - \frac{r^2}{a^2}\right)e^{-\frac{r}{a}} \end{aligned}$$

so we have a minimum provided

$$a^2 + ar - r^2 > 0$$

This is satisfied as long as

$$r < \frac{a\left(1 + \sqrt{5}\right)}{2}$$

and always if $r < a$. We take $r < a$ in the following, since it allows us to expand in powers of $\frac{r}{a}$, and guarantees that the extremum is a minimum.

Notice that to first order in $\frac{r}{a}$, $e^{-\frac{r}{a}} \approx 1 - \frac{r}{a}$ and the circular orbits lie at approximately

$$\begin{aligned}
\frac{L^2}{\mu} &= kr\left(1+\frac{r}{a}\right)e^{-\frac{r}{a}} \\
&\approx kr\left(1+\frac{r}{a}\right)\left(1-\frac{r}{a}\right) \\
&= kr\left(1+\left(\frac{r}{a}\right)^2\right) \\
&\approx kr
\end{aligned}$$

the same value as for the Newtonian potential.

3 Nearly circular orbits

Now consider the precession of nearly circular orbits in the Yukawa potential,

$$V(r) = -\frac{k}{r}e^{-\frac{r}{a}}$$

We know from our general results that the conserved energy and angular momentum are given by

$$\begin{aligned}
E &= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r}e^{-\frac{r}{a}} \\
L &= \mu r^2\dot{\varphi}
\end{aligned}$$

For circular orbits, $r = r_0$, so

$$\begin{aligned}
E &= \frac{L^2}{2\mu r_0^2} - \frac{k}{r_0}e^{-\frac{r_0}{a}} \\
L &= \mu r_0^2\dot{\varphi}_0
\end{aligned}$$

Combining with the condition for a minimum,

$$\frac{L^2}{\mu} = kr\left(1+\frac{r}{a}\right)e^{-\frac{r}{a}}$$

this gives

$$\begin{aligned}
E &= \left(\frac{1}{2r_0^2}kr_0\left(1+\frac{r_0}{a}\right) - \frac{k}{r_0}\right)e^{-\frac{r_0}{a}} \\
&= -\frac{k}{2r_0}\left(1-\frac{r_0}{a}\right)e^{-\frac{r_0}{a}} \\
L &= \mu r_0^2\dot{\varphi}_0
\end{aligned}$$

Therefore, both L and E are determined by r_0,

$$\begin{aligned}
E &= -\frac{k}{2r_0}\left(1-\frac{r_0}{a}\right)e^{-\frac{r_0}{a}} \\
L &= \sqrt{\mu k r_0\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}}
\end{aligned}$$

or correct to second order,

$$
\begin{aligned}
E &= -\frac{k}{2r_0}\left(1-\frac{r_0}{a}\right)\left(1-\frac{r_0}{a}+\frac{1}{2}\left(\frac{r_0}{a}\right)^2\right) \\
&= -\frac{k}{2r_0}\left(1-\frac{2r_0}{a}+\frac{3}{2}\left(\frac{r_0}{a}\right)^2\right) \\
L &= \sqrt{\mu k a\frac{r_0}{a}}\sqrt{\left(1-\frac{1}{2}\left(\frac{r_0}{a}\right)^2\right)} \\
&= \sqrt{\mu k r_0}\left(1-\frac{1}{4}\left(\frac{r_0}{a}\right)^2\right)
\end{aligned}
$$

This also gives us the frequency of the circular orbit,

$$
\begin{aligned}
\omega_0 &= \dot{\varphi}_0 \\
&= \frac{\sqrt{\mu k r_0}\left(1-\frac{1}{4}\left(\frac{r_0}{a}\right)^2\right)}{\mu r_0^2} \\
&= \sqrt{\frac{k}{\mu r_0^3}}\left(1-\frac{1}{4}\left(\frac{r_0}{a}\right)^2\right)
\end{aligned}
$$

Now suppose we give the system slightly higher energy by instantaneously increasing the angular momentum by δ. Then

$$
\begin{aligned}
L &= \mu r_0^2\dot{\varphi}_0+\delta \\
&= L_0+\delta \\
E &= \frac{(L_0+\delta)^2}{2\mu r_0^2}-\frac{k}{r_0}e^{-\frac{r_0}{a}} \\
&= \frac{L_0^2}{2\mu r_0^2}-\frac{k}{r_0}e^{-\frac{r_0}{a}}+\frac{2L_0\delta+\delta^2}{2\mu r_0^2} \\
&= E_0+\frac{2L_0\delta+\delta^2}{2\mu r_0^2}
\end{aligned}
$$

Therefore, at general r, φ,

$$
E_0+\frac{2L_0\delta+\delta^2}{2\mu r_0^2} = \frac{1}{2}\mu\dot{r}^2+\frac{L^2}{2\mu r^2}-\frac{k}{r}e^{-\frac{r}{a}}
$$

The minimum of the potential is still given by

$$
\frac{L^2}{\mu} = k r_1\left(1+\frac{r_1}{a}\right)e^{-\frac{r_1}{a}}
$$

Expanding about r_1, so that $r = r_1+\varepsilon$ with $\varepsilon \ll r_1 < a$

$$
\begin{aligned}
E_0+\frac{2L_0\delta+\delta^2}{2\mu r_0^2} &= \frac{1}{2}\mu\dot{r}^2+\frac{L^2}{2\mu r^2}-\frac{k}{r}e^{-\frac{r}{a}} \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2+\frac{L^2}{2\mu(r_1+\varepsilon)^2}-\frac{k}{r_1+\varepsilon}e^{-\frac{r_1+\varepsilon}{a}} \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2+\frac{L^2}{2\mu r_1^2\left(1+\frac{\varepsilon}{r_1}\right)^2}-\frac{k}{r_1\left(1+\frac{\varepsilon}{r_1}\right)}e^{-\frac{r_1}{a}\left(1+\frac{\varepsilon}{r_1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{kr_1\left(1+\frac{r_1}{a}\right)e^{-\frac{r_1}{a}}}{2r_1^2\left(1+\frac{\varepsilon}{r_1}\right)^2} - \frac{k}{r_1\left(1+\frac{\varepsilon}{r_1}\right)}e^{-\frac{r_1}{a}\left(1+\frac{\varepsilon}{r_1}\right)} \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{ke^{-\frac{r_1}{a}}}{r_1}\left(\frac{\left(1+\frac{r_1}{a}\right)}{2\left(1+\frac{\varepsilon}{r_1}\right)^2} - \frac{1}{\left(1+\frac{\varepsilon}{r_1}\right)}e^{-\frac{r_1}{a}\frac{\varepsilon}{r_1}}\right) \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{ke^{-\frac{r_1}{a}}}{r_1}\frac{1}{2}\left(1+\frac{r_1}{a}\right)\left(1-\frac{2\varepsilon}{r_1}+3\left(\frac{\varepsilon}{r_1}\right)^2\right) \\
&\quad -\frac{ke^{-\frac{r_1}{a}}}{r_1}\left(1-\frac{\varepsilon}{r_1}+\left(\frac{\varepsilon}{r_1}\right)^2\right)\left(1-\frac{r_1}{a}\frac{\varepsilon}{r_1}+\frac{1}{2}\left(\frac{r_1}{a}\frac{\varepsilon}{r_1}\right)^2\right) \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{ke^{-\frac{r_1}{a}}}{r_1}\frac{1}{2}\left(\left(1+\frac{r_1}{a}\right)-2\left(1+\frac{r_1}{a}\right)\frac{\varepsilon}{r_1}+3\left(\frac{\varepsilon}{r_1}\right)^2\right) \\
&\quad -\frac{ke^{-\frac{r_1}{a}}}{r_1}\left(1-\left(1+\frac{r_1}{a}\right)\frac{\varepsilon}{r_1}+\left(1+\frac{r_1}{a}+\frac{1}{2}\frac{r_1^2}{a^2}\right)\left(\frac{\varepsilon}{r_1}\right)^2\right) \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{ke^{-\frac{r_1}{a}}}{r_1}\left(-\frac{1}{2}\left(1-\frac{r_1}{a}\right)+\left(1+\frac{r_1}{a}-\left(1+\frac{r_1}{a}\right)\right)\frac{\varepsilon}{r_1}+\frac{1}{2}\left(1+\frac{r_1}{a}-\frac{r_1^2}{a^2}\right)\left(\frac{\varepsilon}{r_1}\right)^2\right) \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 - \frac{1}{2}\left(1-\frac{r_1}{a}\right)\frac{ke^{-\frac{r_1}{a}}}{r_1} + \frac{1}{2}\left(1+\frac{r_1}{a}-\frac{r_1^2}{a^2}\right)\frac{ke^{-\frac{r_1}{a}}}{r_1}\left(\frac{\varepsilon}{r_1}\right)^2
\end{aligned}
$$

so that

$$
E_0 + \frac{2L_0\delta+\delta^2}{2\mu r_0^2} + \frac{1}{2}\left(1-\frac{r_1}{a}\right)\frac{ke^{-\frac{r_1}{a}}}{r_1} = \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{1}{2}\left(1+\frac{r_1}{a}-\frac{r_1^2}{a^2}\right)\frac{ke^{-\frac{r_1}{a}}}{r_1}\left(\frac{\varepsilon}{r_1}\right)^2
$$

This has the general form

$$
E_\delta = \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{1}{2}\kappa^2\varepsilon^2
$$

and this is the energy of a simple harmonic oscillator, oscillating around a point at r_1 with angular frequency

$$
\omega_1 = \sqrt{\left(1+\frac{r_1}{a}-\frac{r_1^2}{a^2}\right)\frac{ke^{-\frac{r_1}{a}}}{\mu r_1^3}}
$$

while the orbital frequency is

$$
\begin{aligned}
L &= \sqrt{\mu k r_1\left(1+\frac{r_1}{a}\right)e^{-\frac{r_1}{a}}} \\
\dot{\varphi}_1 &= \frac{1}{\mu r^2}\sqrt{\mu k r_1\left(1+\frac{r_1}{a}\right)e^{-\frac{r_1}{a}}} \\
&\approx \frac{1}{\mu r_1^2}\sqrt{\mu k r_1\left(1+\frac{r_1}{a}\right)e^{-\frac{r_1}{a}}}
\end{aligned}
$$

The ratio of frequencies is

$$
\frac{\omega_1}{\dot{\varphi}_1} \approx \sqrt{\frac{\left(1+\frac{r_1}{a}-\frac{r_1^2}{a^2}\right)\frac{ke^{-\frac{r_1}{a}}}{\mu r_1^3}\mu^2 r_1^4}{\mu k r_1\left(1+\frac{r_1}{a}\right)e^{-\frac{r_1}{a}}}}
$$

$$
\begin{aligned}
&= \sqrt{\frac{1+\frac{r_1}{a}-\frac{r_1^2}{a^2}}{1+\frac{r_1}{a}}} \\
&\approx \sqrt{\left(1+\frac{r_1}{a}-\frac{r_1^2}{a^2}\right)\left(1-\frac{r_1}{a}+\left(\frac{r_1}{a}\right)^2\right)} \\
&\approx \sqrt{1-\frac{r_1^2}{a^2}} \\
&\approx 1-\frac{1}{2}\frac{r_1^2}{a^2}
\end{aligned}
$$

and this varies continuously with r_1 and is therefore generically irrational, so the orbit doesn't close. The oscillation frequency in r is less than the orbital frequency, so the perigee advances. The advance per orbit is $\pi\frac{r_1^2}{a^2}$.