Further developments

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1 Conservation and cyclic coordinates

From the relationship between the Lagrangian and the Hamiltonian,

$$H = p_i \dot{x}_i - L$$

we see that if a coordinate is cyclic in the Lagrangian it is also cyclic in the Hamiltonian,

$$\frac{\partial H}{\partial x_i} = -\frac{\partial L}{\partial x_i}$$

When a coordinate x_i is cyclic then the corresponding Hamilton equation reads

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = 0$$

and the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

is conserved, so the relationship between cyclic coordinates and conserved quantities still holds.

Hamilton's equations show that we also have a corresponding statement about momentum. Suppose some momentum, p_i, is cyclic in the Hamiltonian,

$$\frac{\partial H}{\partial p_i} = 0$$

Then from Hamilton's equations we immediately have

$$\dot{x}_i = 0$$

so that the coordinate x_i is a constant of the motion.

Suppose we have a cyclic coordinate, say x_n. Then the conserved momentum takes its initial value, p_{n0}, and the Hamiltonian is

$$H = H\left(x_1, \ldots x_{n-1}; p_1, \ldots p_{n-1}, p_{n0}\right)$$

and therefore immediately becomes a function of $2\left(n-1\right)$ variables. This is simpler than the Lagrangian case, where constancy of p_n makes no immediate simplification of the Lagrangian.

Consider the time derivative of the Hamiltonian,

$$\begin{aligned}
\frac{dH}{dt} &= \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q^i}\dot{q}^i + \frac{\partial H}{\partial t} \\
&= \dot{q}^i \dot{p}_i - \dot{p}_i \dot{q}^i + \frac{\partial H}{\partial t} \\
&= \frac{\partial H}{\partial t}
\end{aligned}$$

so the Hamiltonian is conserved if it does not explicitly depend on time.

Example 1: As a simple example, consider the 2-dimensional Kepler problem, with Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GM}{r}$$

with θ cyclic. The momenta are easily seen to be

$$\begin{aligned} p_r &= m\dot{r} \\ p_\theta &= mr^2\dot{\theta} \end{aligned}$$

so the Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GM}{r}$$

Here θ is cyclic to the conserved momentum p_θ is constant.

Example 2: Let a mass, m, free to move in one direction, experience a Hooke's law restoring force, $F = -kx$. Solve Hamilton's equations and study the motion of system in phase space. The Lagrangian for this system is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \end{aligned}$$

The conjugate momentum is just

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

so the Hamiltonian is

$$\begin{aligned} H &= p\dot{x} - L \\ &= \frac{p^2}{m} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2 \\ &= \frac{1}{2m}\left(p^2 + m^2\omega^2x^2\right) \end{aligned}$$

We may write this in terms of $\xi^A = (x, p)$ as

$$H = \frac{1}{2m}H_{AB}\xi^A\xi^B$$

where

$$H_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & m^2\omega^2 \end{pmatrix}$$

Since $\frac{\partial H}{\partial t} = 0$, $E = H$ is a constant of the motion. We see immediately that the solution is an ellipse in phase space, $E = \frac{1}{2m}\left(p^2 + m^2\omega^2x^2\right)$, or

$$\frac{1}{2mE}p^2 + \frac{m\omega^2}{2E}x^2 = 1$$

The solution with initial conditions $x(0) = x_0, p(0) = p_0$ has $E = \frac{1}{2m}\left(p_0^2 + m^2\omega^2x_0^2\right)$

$$\begin{aligned} x &= \sqrt{\frac{2mE}{m^2\omega^2}}\sin\lambda \\ p &= \sqrt{2mE}\cos\lambda \end{aligned}$$

where λ is some function of time. To find λ, we look at one of Hamilton's equations,

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p} \\
&= \frac{p}{m} \\
\sqrt{\frac{2mE}{m^2\omega^2}}\dot{\lambda}\cos\lambda &= \frac{\sqrt{2mE}}{m}\cos\lambda \\
\dot{\lambda} &= \omega \\
\lambda &= \omega t + \varphi_0
\end{aligned}$$

and therefore

$$\begin{aligned}
x &= \sqrt{\frac{2mE}{m^2\omega^2}}\sin(\omega t + \varphi_0) \\
p &= \sqrt{2mE}\cos(\omega t + \varphi_0)
\end{aligned}$$

where $\sqrt{\frac{2mE}{m^2\omega^2}}\cos\varphi_0 = x_0$ and $p_0 = \sqrt{2mE}\sin\varphi_0$, or,

$$\begin{aligned}
\cos\varphi_0 &= \frac{m\omega x_0}{\sqrt{2mE}} \\
\sin\varphi_0 &= \frac{p_0}{\sqrt{2mE}}
\end{aligned}$$

2 The symplectic form

2.1 Writing Hamilton's equations with unified variables

In order to fully appreciate the power and uses of Hamiltonian mechanics, we develop some formal properties. First, we write Hamilton's equations,

$$\begin{aligned}
\dot{x}_k &= \frac{\partial H}{\partial p_k} \\
\dot{p}_k &= -\frac{\partial H}{\partial x_k}
\end{aligned}$$

for $k = 1, \ldots, n$, in a different way. Define a unified name for our $2n$ coordinates,

$$\xi_A = (x_i, p_j)$$

for $A = 1, \ldots, 2n$. That is, more explicitly, for $i = 1, \ldots, n$,

$$\begin{aligned}
\xi_i &= x_i \\
\xi_{n+i} &= p_i
\end{aligned}$$

We may immediately write the left side of both of Hamilton's equations at once as

$$\dot{\xi}_A = (\dot{x}_i, \dot{p}_j)$$

The right side of the equations involves all of the derivatives,

$$\frac{\partial H}{\partial \xi_A} = \left(\frac{\partial H}{\partial x_i}, \frac{\partial H}{\partial p_j}\right) \tag{1}$$

but there is a difference of a minus sign between the two equations and the interchange of x_i and p_i. We incorporate this by introducing a matrix called the *symplectic form*,

$$\Omega_{AB} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

where $[\mathbf{1}]_{ij} = \delta_{ij}$ is the $n \times n$ identity matrix. Then, using the summation convention, Hamilton's equations take the form of a single expression,

$$\dot{\xi}_A = \Omega_{AB} \frac{\partial H}{\partial \xi_B}$$

We may check this by writing it out explicitly,

$$\begin{aligned} \begin{pmatrix} \dot{x}_i \\ \dot{p}_j \end{pmatrix} &= \begin{pmatrix} 0 & \delta_{in} \\ -\delta_{jm} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x_m} \\ \frac{\partial H}{\partial p_n} \end{pmatrix} \\ &= \begin{pmatrix} \delta_{im} \frac{\partial H}{\partial p_m} \\ -\delta_{jn} \frac{\partial H}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial x_j} \end{pmatrix} \end{aligned}$$

Example: Coupled pendula For the example of two simple pendula coupled by a spring, the Hamiltonian for small angles is

$$H = \frac{1}{2ml^2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}kl^2\left(\theta_1 - \theta_2\right)^2 + \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right)$$

and we set $\xi_1 = \theta_1, \xi_2 = \theta_2, \xi_3 = p_1$ and $\xi_4 = p_2$. In terms of these, the Hamiltonian may be written as a symmetric quadratic form

$$\begin{aligned} H &= \frac{1}{2} H_{AB} \xi_A \xi_B \\ H_{AB} &= \begin{pmatrix} kl^2 + mgl & -kl^2 & 0 & 0 \\ -kl^2 & kl^2 + mgl & 0 & 0 \\ 0 & 0 & \frac{1}{ml^2} & 0 \\ 0 & 0 & 0 & \frac{1}{ml^2} \end{pmatrix} \end{aligned}$$

with

$$\frac{\partial}{\partial \xi_C} H = \frac{1}{2} H_{AB} \delta_{AC} \xi_B + \frac{1}{2} H_{AB} \xi_A \delta_{BC} = H_{CB} \xi_B$$

Hamilton's equations are then

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} kl^2 + mgl & -kl^2 & 0 & 0 \\ -kl^2 & kl^2 + mgl & 0 & 0 \\ 0 & 0 & \frac{1}{ml^2} & 0 \\ 0 & 0 & 0 & \frac{1}{ml^2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \frac{1}{ml^2} & 0 \\ 0 & 0 & 0 & \frac{1}{ml^2} \\ -kl^2 - mgl & kl^2 & 0 & 0 \\ kl^2 & -kl^2 - mgl & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{ml^2}\xi_3 \\ \frac{1}{ml^2}\xi_4 \\ -kl^2\xi_1 - mgl\xi_1 + kl^2\xi_2 \\ -kl^2\xi_2 - mgl\xi_2 + kl^2\xi_1 \end{pmatrix} \end{aligned}$$

so that

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{ml^2}\xi_3 \\ \frac{1}{ml^2}\xi_4 \\ -kl^2(\xi_1 - \xi_2) - mgl\xi_1 \\ kl^2(\xi_1 - \xi_2) - mgl\xi_2 \end{pmatrix}$$

as expected.

2.2 Properties of the symplectic form

We note a number of important properties of the symplectic form. First, it is antisymmetric,

$$\begin{aligned} \Omega^t &= -\Omega \\ \Omega_{AB} &= -\Omega_{BA} \end{aligned}$$

and it squares to minus the $2n$-dimensional identity,

$$\begin{aligned} \Omega^2 &= -\mathbf{1} \\ \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} &= \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \\ &= -\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \end{aligned}$$

We also have

$$\Omega^t = \Omega^{-1}$$

since $\Omega^t = -\Omega$, and therefore $\Omega\Omega^t = \Omega(-\Omega) = -\Omega^2 = \mathbf{1}$. Since all components of Ω_{AB} are constant, it is also true that

$$\partial_A \Omega_{BC} = \frac{\partial}{\partial \xi_A}\Omega_{BC} = 0$$

This last condition does not hold in every basis, however.

The defining properties of the symplectic form, necessary and sufficient to guarantee that it has the properties we require for Hamiltonian mechanics are that it be a $2n \times 2n$ matrix satisfying two properties at each point of phase space:

1. $\Omega^2 = -\mathbf{1}$
2. $\partial_A \Omega_{BC} + \partial_B \Omega_{CA} + \partial_C \Omega_{AB} = 0$

The first of these is enough for there to exist a change of basis so that $\Omega_{AB} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ at any given point, while the vanishing combination of derivatives insures that this may be done at *every* point of phase space.

2.3 Change of coordinates

Consider what happens to Hamilton's equations if we want to change to a new set of phase space coordinates, $\chi^A = \chi^A(\xi)$. Let the inverse transformation be $\xi^A(\chi)$. The time derivatives become

$$\frac{d\xi^A}{dt} = \frac{\partial \xi^A}{\partial \chi^B}\frac{d\chi^B}{dt}$$

while the right side of Hamilton's equation becomes

$$\Omega^{AB}\frac{\partial H}{\partial \xi^B} = \Omega^{AB}\frac{\partial \chi^C}{\partial \xi^B}\frac{\partial H}{\partial \chi^C}$$

Equating these expressions,

$$\frac{\partial \xi^A}{\partial \chi^B}\frac{d\chi^B}{dt} = \Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D}$$

we multiply by the Jacobian matrix, $\frac{\partial \chi^C}{\partial \xi^A}$ to get

$$\begin{aligned}
\frac{\partial \chi^C}{\partial \xi^A}\frac{\partial \xi^A}{\partial \chi^B}\frac{d\chi^B}{dt} &= \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D} \\
\delta^C_B\frac{d\chi^B}{dt} &= \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D}
\end{aligned}$$

and finally

$$\frac{d\chi^C}{dt} = \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D}$$

Defining the symplectic form in the new coordinate system,

$$\tilde{\Omega}^{CD} \equiv \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}$$

we see that Hamilton's equations are entirely the same if the transformation leaves the symplectic form invariant,

$$\tilde{\Omega}^{CD} = \Omega^{CD}$$

Any linear transformation $M^A{}_B$ leaving the symplectic form invariant,

$$\Omega^{AB} \equiv M^A{}_C M^B{}_D \Omega^{CD}$$

is called a symplectic transformation. Coordinate transformations which are symplectic transformations at each point are called *canonical.* Therefore those functions $\chi^A(\xi)$ satisfying

$$\Omega^{CD} \equiv \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}$$

are canonical transformations. Canonical transformations preserve Hamilton's equations.

2.4 Poincaré sections

The phase space description of classical systems are equivalent to the configuration space solutions and are often easier to interpret because more information is displayed at once. The price we pay for this is the doubled dimension – paths rapidly become difficult to plot. To offset this problem, we can use *Poincaré sections* – projections of the phase space plot onto subspaces that cut across the trajectories. Sometimes the patterns that occur on Poincaré sections show that the motion is confined to specific regions of phase space, even when the motion never repeats itself. These techniques allow us to study systems that are *chaotic*, meaning that the phase space paths through nearby points diverge rapidly. See the Wikipedia page on Chaos Theory. For more detail, read Gleick, Chaos: Making a New Science.