# Scattering

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## General formalism

Unbounded orbits are typically encountered in scattering experiments. A beam of particles is directed at a target, and the resulting interactions of the beam with the target particles scatters the beam particles in all directions, with a probability that depends on the forces. The target particles may be molecules, atoms, nuclei, or other fundamental particles, and the beam is generally electrons, protons or heavy nuclei.

The variable measured is called the differential cross-section, $d\sigma$, defined as

$$d\sigma = \frac{d\sigma}{d\Omega}d\Omega = \frac{number\ of\ particles\ scattered\ into\ solid\ angle\ d\Omega\ per\ unit\ time}{incident\ intensity\ (particles\ per\ unit\ area\ per\ unit\ time)}$$

with units of area. The solid angle, $d\Omega$, is given by

$$d\Omega = \sin\theta d\Theta d\Phi$$

where our use of upper case Greek letters distinguishes the center of mass frame $(\Theta, \Phi)$ from the lab frame $(\theta, \varphi)$. We take the $z$-axis as the direction of the incident beam, so that deviations from that direction are given by $\Theta$. Since scattering by central forces cannot depend on azimuthal angle, we may integrate over $\varphi$ and look at the probability for scattering into an annulus at angle $\Theta$, of solid angle

$$d\Omega = 2\pi \sin\Theta d\Theta$$

Since the approaching beam of particles (of intensity, $I$) is generally moving at a fixed velocity, the total energy, $E = \frac{1}{2}mv_0^2$, of the scattering is the same for all ecounters. It is therefore only the total angular momentum, $l$, that determines the angle of scatter. We can find $l$ by extending the initial beam particle trajectory past the target. The distance of closest approach of this line is called the impact parameter, $s$, and the angular momentum is just

$$\begin{aligned} l &= mv_0 s \\ &= s\sqrt{2mE} \end{aligned}$$

If we let $N(\Theta)\, d\Omega$ be the number of particles scattered between $\Theta$ and $\Theta + d\Theta$,

$$N(\Theta)\, d\Omega = 2\pi I \left(\frac{d\sigma}{d\Omega}\right) \sin\Theta \left|d\Theta\right|$$

this must equal the number with impact parameter between the corresponding $s$ and $s + ds$,

$$N(\Theta)\, d\Omega = 2\pi s I \left|ds\right|$$

we we need only find the relationship between impact parameter and scattering angle,

$$s(\Theta, E)$$

Then we have

$$\begin{aligned} 2\pi I \left(\frac{d\sigma}{d\Omega}\right) \sin\Theta\,|d\Theta| &= 2\pi s I\,|ds| \\ \frac{d\sigma}{d\Omega} &= \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \end{aligned}$$

From our discussion of central forces, we have the angle as an integral over $r$,

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2\mu E}{L_\varphi^2} - \frac{1}{r^2} - \frac{2\mu V}{L_\varphi^2}}}$$

where $r = r_0$ when $\varphi = \varphi_0$. Suppose $\mu \approx m$, and let $\Psi_0 = 0$ when $r = r_{min}$, the actual point of closest approach. Then if we integrate out to $r = \infty$, we see that $2\Psi$ is the complement to $\Theta$,

$$\Theta = \pi - 2\Psi$$

with

$$\begin{aligned} \Psi &= \int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2\mu E}{L_\varphi^2} - \frac{1}{r^2} - \frac{2\mu V}{L_\varphi^2}}} \\ &= \int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{s^2 2mE} - \frac{1}{r^2} - \frac{2mV}{s^2 2mE}}} \\ &= \int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{s^2}\left(1 - \frac{V}{E}\right) - \frac{1}{r^2}}} \\ &= \int_{r_{min}}^{\infty} \frac{s\,dr}{r\sqrt{r^2\left(1 - \frac{V}{E}\right) - s^2}} \end{aligned}$$

Now set $r = \frac{1}{u}$,

$$\begin{aligned} \Psi &= \int_{r_{min}}^{\infty} \frac{-\frac{1}{u^2} s u\,du}{\sqrt{\frac{1}{u^2}\left(1 - \frac{V}{E}\right) - s^2}} \\ &= -s\int_{r_{min}}^{\infty} \frac{du}{\sqrt{1 - \frac{V(u)}{E} - s^2 u^2}} \end{aligned}$$

# Coulomb Scattering

As an important example, consider the case of repulsive Coulomb scattering,

$$\begin{aligned} f(r) &= \frac{ZZ'e^2}{r^2} \\ &= -\frac{k}{r^2} \end{aligned}$$

Then we have hyperbolic orbits,

$$r = \frac{|A|\left(\varepsilon^2 - 1\right)}{1 + \varepsilon\cos\varphi}$$

where, substituting Coulomb constants for the gravitational constants, the eccentricity is given by

$$\begin{aligned}
\varepsilon &= \sqrt{1+\frac{2EL_\varphi^2}{\left(ZZ'e^2\right)^2 m}} \\
&= \sqrt{1+\frac{2E\left(s^2 2mE\right)}{\left(ZZ'e^2\right)^2 m}} \\
&= \sqrt{1+\left(\frac{2Es}{ZZ'e^2}\right)^2}
\end{aligned}$$

In this case, the angle $\Psi$ is just the angle between the zero and maximum of the denominator in $r$. Since the maximum occurs when $\Psi = 0$, we have

$$\begin{aligned}
1+\varepsilon\cos\Psi_{min} &= 1+e \\
1+\varepsilon\cos\Psi_{max} &= 0 \\
\Psi &= \Psi_{max}-\Psi_{min} \\
&= \cos^{-1}\left(-\frac{1}{\varepsilon}\right)-0 \\
&= \cos^{-1}\left(\frac{1}{\varepsilon}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Theta &= \pi-2\Psi \\
&= \pi-2\cos^{-1}\left(\frac{1}{\varepsilon}\right) \\
\frac{\Theta}{2}-\frac{\pi}{2} &= \cos^{-1}\left(\frac{1}{\varepsilon}\right) \\
\cos\left(\frac{\Theta}{2}-\frac{\pi}{2}\right) &= \frac{1}{\varepsilon} \\
\sin\frac{\Theta}{2} &= \frac{1}{\varepsilon}
\end{aligned}$$

Now solving for $s(\Theta)$,

$$\begin{aligned}
\varepsilon &= \sqrt{1+\left(\frac{2Es}{ZZ'e^2}\right)^2} \\
\varepsilon^2-1 &= \left(\frac{2Es}{ZZ'e^2}\right)^2 \\
s &= \frac{ZZ'e^2}{2E}\sqrt{\varepsilon^2-1} \\
&= \frac{ZZ'e^2}{2E}\sqrt{\frac{1}{\sin^2\frac{\Theta}{2}}-1} \\
s &= \frac{ZZ'e^2}{2E}\frac{\cos\frac{\Theta}{2}}{\sin\frac{\Theta}{2}}
\end{aligned}$$

so that we have simply

$$s = \frac{ZZ'e^2}{2E}\cot\frac{\Theta}{2}$$

$$\frac{ds}{d\Theta} = -\frac{ZZ'e^2}{4E\sin^2\frac{\Theta}{2}}$$

and substituting into the differential cross section,

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \\
&= \frac{\left(\frac{ZZ'e^2}{2E}\frac{\cos\frac{\Theta}{2}}{\sin\frac{\Theta}{2}}\right)}{2\cos\frac{\Theta}{2}\sin\frac{\Theta}{2}}\frac{ZZ'e^2}{4E\sin^2\frac{\Theta}{2}} \\
&= \frac{Z^2Z'^2e^4}{16E^2\sin^4\frac{\Theta}{2}}
\end{aligned}$$

The result is the Rutherford cross section

$$\frac{d\sigma}{d\Omega} = \frac{Z^2Z'^2e^4}{16E^2\sin^4\frac{\Theta}{2}}$$

which allowed for the discovery of the nucleus in the early 20th century, even though the calculation properly requires a quantum treatment. It is a happy coincidence that the full quantum treatment still gives the characteristic $\frac{1}{\sin^4\frac{\Theta}{2}}$ behavior.