Rigid Body Dynamics

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1 Non-inertial frames of reference

So far we have formulated classical mechanics in inertial frames of reference, i.e., those vector bases in which Newton's second law holds (we have also allowed general coordinates, in which the Euler-Lagrange equations hold). However, it is sometimes useful to use non-inertial frames, and particularly when a system is rotating. When we affix an orthonormal frame to the surface of Earth, for example, that frame rotates with Earth's motion and is therefore non-inertial. The effect of this is to add terms to the acceleration due to the acceleration of the reference frame. Typically, these terms can be brought to the force side of the equation, giving rise to the idea of fictitious forces – centrifugal force and the Coriolis force are examples.

Here we concern ourselves with rotating frames of reference.

2 Rotating frames of reference

2.1 Relating rates of change in inertial and rotating systems

It is fairly easy to include the effect of a rotating vector basis. Consider the change, $d\mathbf{b}$, of some physical quantity describing a rotating body. We write this in two different reference frames, one inertial and one rotating with the body. The difference between these will be the change due to the rotation,

$$(d\mathbf{b})_{inertial} = (d\mathbf{b})_{body} + (d\mathbf{b})_{rot}$$

Now consider an infinitesimal rotation. We showed that the transformation matrix must have the form

$$O(d\theta, \hat{\mathbf{n}}) = 1 + d\theta\hat{\mathbf{n}} \cdot \mathbf{J}$$

where

$$[J_i]_{jk} = \varepsilon_{ijk}$$

Using this form of $\mathbf{J}$, we may write

$$[O(d\theta, \hat{\mathbf{n}})]_{jk} = \delta_{jk} + d\theta n_i \varepsilon_{ijk}$$

We must establish the direction of this rotation. Suppose $\hat{\mathbf{n}}$ is in the z-direction, $n_i = (0,0,1)$. Then acting on a vector in the xy-plane, say $[\mathbf{i}]_i = (1,0,0)$, we have

$$\begin{aligned} [O(d\theta, \hat{\mathbf{n}})]_{jk} i_k &= (\delta_{jk} + d\theta n_i \varepsilon_{ijk}) i_k \\ &= (i_j + d\theta \varepsilon_{3j1}) \\ &= (1,0,0) + d\theta (0,-1,0) \end{aligned}$$

since ε_{3j1} must have $j = 2$ to be nonzero, and $\varepsilon_{321} = -1$. The vector acquires a negative y-component, and has therefore rotated clockwise. A counterclockwise (positive) rotation is therefore given by acting with

$$O(d\theta, \hat{\mathbf{n}}) = 1 - d\theta\hat{\mathbf{n}} \cdot \mathbf{J}$$

Suppose a vector at time t, $\mathbf{b}(t)$ is fixed in a body which rotates with angular velocity $\boldsymbol{\omega} = \frac{d\theta}{dt}\mathbf{n}$. Then after a time dt it will have rotated through an angle $d\theta = \omega dt$, so that at time $t + dt$ the vector is

$$\mathbf{b}(t+dt) = O(d\theta, \hat{\mathbf{n}})\,\mathbf{b}(t)$$

In components,

$$\begin{aligned} b_j(t+dt) &= (\delta_{jk} - d\theta n_i \varepsilon_{ijk})\, b_k(t) \\ &= \delta_{jk} b_k(t) - d\theta n_i \varepsilon_{ijk} b_k(t) \\ &= b_j(t) - d\theta n_i \varepsilon_{ijk} b_k(t) \\ &= b_j(t) - d\theta\,(\varepsilon_{kij} b_k(t)\, n_i) \end{aligned}$$

Therefore, returning to vector notation,

$$\mathbf{b}(t+dt) - \mathbf{b}(t) = -d\theta\mathbf{b}(t) \times \mathbf{n}$$

Dividing by dt we get the rate of change,

$$\frac{d\mathbf{b}(t)}{dt} = \boldsymbol{\omega} \times \mathbf{b}(t)$$

If, instead of remaining fixed in the rotating system, $\mathbf{b}(t)$ moves relative to the rotating body, its rate of change is the sum of this change and the rate of change due to rotation,

$$\left(\frac{d\mathbf{b}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{b}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{b}(t)$$

and since $\mathbf{b}(t)$ is arbitrary, we can make the operator identification

$$\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times$$

2.2 Dynamics in a rotating frame of reference

Consider two frames of reference, an inertial frame, and a rotating frame whose origin remains at the origin of the inertial frame. Let $\mathbf{r}(t)$ be the position vector of a particle in the rotating frame of reference. Then the velocity of the particle in an inertial frame, $\mathbf{v}_{inertial}$, and the velocity in the rotating frame, $\mathbf{v}_{body}$, are related by

$$\begin{aligned} \left(\frac{d\mathbf{r}}{dt}\right)_{inertial} &= \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{r} \\ \mathbf{v}_{inertial} &= \mathbf{v}_{body} + \boldsymbol{\omega} \times \mathbf{r} \end{aligned}$$

To find the acceleration, we apply the operator again,

$$\begin{aligned} \frac{d\mathbf{v}_{inertial}}{dt} &= \left(\frac{d}{dt} + \boldsymbol{\omega}\times\right)(\mathbf{v}_{body} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \frac{d(\mathbf{v}_{body} + \boldsymbol{\omega} \times \mathbf{r})}{dt} + \boldsymbol{\omega} \times (\mathbf{v}_{body} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \left(\frac{d\mathbf{v}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \left(\frac{d\mathbf{v}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_{body} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \end{aligned}$$

The accelerations are therefore related by

$$\mathbf{a}_{inertial} = \mathbf{a}_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_{body} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Since Newton's second law holds in the inertial frame, we have

$$\mathbf{F} = m\mathbf{a}_{inertial}$$

where $\mathbf{F}$ refers to any applied forces. Therefore, bringing the extra terms to the left,

$$\mathbf{F} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \mathbf{v}_{body} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\mathbf{a}_{body}$$

This is the Coriolis theorem. We consider each term.

The first

$$-m\frac{d\boldsymbol{\omega}}{dt}$$

applies only if the rate of rotation is changing. The direction makes sense, because if the angular velocity is increasing, then $\frac{d\boldsymbol{\omega}}{dt}$ is in the direction of the rotation and the inertia of the particle will resist this change. The effective force is therefore in the opposite direction.

The second term

$$-2m\boldsymbol{\omega} \times \mathbf{v}_{body}$$

is called the Coriolis force. Notice that it is greatest if the velocity is perpendicular to the axis of rotation. This corresponds to motion which, for positive $\mathbf{v}_{body}$, moves the particle further from the axis of rotation. Since the velocity required to stay above a point on a rotating body increases with increasing distance from the axis, the particle will be moving too slow to keep up. It therefore seems that a force is acting in the direction opposite to the direction of rotation. For example, consider a particle at Earth's equator which is gaining altitude. Since Earth rotates from west to east, the rising particle will fall behind and therefore seem to accelerate from toward the west.

The final term

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

is the familiar centrifugal force (arising from centripetal acceleration). For Earth's rotation, $\boldsymbol{\omega} \times \mathbf{r}$ is the direction of the velocity of a body rotating with Earth, and direction of the centrifugal force is therefore directly away from the axis of rotation. The effect is due to the tendency of the body to move in a straight line in the inertial frame, hence away from the axis. For a particle at the equator, the centrifugal force is directed radially outward, opposing the force of gravity. The net acceleration due to gravity and the centrifugal acceleration is therefore,

$$\begin{aligned} g_{eff} &= g - \omega^2 r \\ &= 9.8 - \left(7.29 \times 10^{-5}\right)^2 \times 6.38 \times 10^6 \\ &= 9.8 - .0339 \\ &= g\left(1 - .035\right) \end{aligned}$$

so that the gravitational attraction is reduced by about 3.5%. Since the effect is absent near the poles, Earth is not a perfect sphere, but has an equatorial bulge.

3 Moment of Inertia

Fix an arbitrary inertial frame of reference, and consider a rigid body. onsider the total torque on the body. The torque on the i^{th} particle due to internal forces will be

$$\boldsymbol{\tau}_i = \sum_{j=1}^{N} \mathbf{r}_i \times \mathbf{F}_{ji}$$

where $\mathbf{F}_{ji}$ is the force exerted by the j^{th} particle on the i^{th} particle. The total torque on the body is therefore the double sum,

$$\begin{aligned}
\boldsymbol{\tau}_{internal} &= \sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{r}_i \times \mathbf{F}_{ji} \\
&= \frac{1}{2}\sum_{i<1}^{N}\sum_{j=1}^{N}\left(\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij}\right) \\
&= \frac{1}{2}\sum_{i<1}^{N}\sum_{j=1}^{N}\left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{ji}
\end{aligned}$$

where we use Newton's third law in the last step. However, we assume that the forces between particles within the rigid body are along the line joining the two particles, so we have

$$\mathbf{F}_{ji} = F_{ji}\frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

so all the cross products vanish, and

$$\boldsymbol{\tau}_{internal} = 0$$

Therefore, we consider only external forces acting on the body when we compute the torque.

Now it is easier to work in the continuum limit. Let the density at each point of the body be $\rho(\mathbf{r})$ (for a discrete collection of masses, we may let ρ be a sum of Dirac delta functions and recover the discrete picture). The contribution to the total torque of an external force $d\mathbf{F}(\mathbf{r})$ acting at position $\mathbf{r}$ of the body is

$$d\boldsymbol{\tau} = \mathbf{r} \times d\mathbf{F}(\mathbf{r})$$

and the total follows by integrating this. Substituting for the force using Newton's second law, $d\mathbf{F}(\mathbf{r}) = \frac{d\mathbf{v}}{dt}dm = \frac{d\mathbf{v}}{dt}\rho(\mathbf{r})\,d^3x$ we have

$$\begin{aligned}
\boldsymbol{\tau} &= \int \mathbf{r} \times \frac{d\mathbf{v}}{dt}dm \\
&= \int \rho(\mathbf{r})\left(\mathbf{r} \times \frac{d\mathbf{v}}{dt}\right)d^3x \\
&= \int \rho(\mathbf{r})\left[\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) - \left(\frac{d\mathbf{r}}{dt} \times \mathbf{v}\right)\right]d^3x
\end{aligned}$$

Since $\frac{d\mathbf{r}}{dt} \times \mathbf{v} = \mathbf{v} \times \mathbf{v} = 0$, and the density is independent of time,

$$\boldsymbol{\tau} = \frac{d}{dt}\int \rho(\mathbf{r})(\mathbf{r} \times \mathbf{v})\,d^3x$$

Notice the the right-hand side is just the total angular momentum, since $d\mathbf{L}$ for a small mass element $dm = \rho d^3x$ is $d\mathbf{L} = \rho(\mathbf{r})(\mathbf{r} \times \mathbf{v})$.

Now suppose the body rotates with angular velocity $\boldsymbol{\omega}$. Then the velocity of any point in the body is $\boldsymbol{\omega} \times \mathbf{r}$, so

$$\begin{aligned}
\boldsymbol{\tau} &= \frac{d}{dt}\int \rho(\mathbf{r})(\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}))\,d^3x \\
&= \frac{d}{dt}\int \rho(\mathbf{r})\left(\boldsymbol{\omega}r^2 - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega})\right)d^3x
\end{aligned}$$

We would like to separate the properties intrinsic to the rigid body from those dependent on its motion. To do this, we extract $\boldsymbol{\omega}$ from the integral above, but this required index notation. Write the equation in components,

$$\begin{aligned}
\tau_i &= \int \rho(\mathbf{r}) \left(\omega_i r^2 - r_i r_j \omega_j\right) d^3x \\
&= \int \rho(\mathbf{r}) \left(\omega_j \delta_{ij} r^2 - r_i r_j \omega_j\right) d^3x \\
&= \omega_j \int \rho(\mathbf{r}) \left(\delta_{ij} r^2 - r_i r_j\right) d^3x
\end{aligned}$$

Notice how the use of dummy indices and the Kronecker delta allows us to get the same index on ω_j in both terms so that we can bring it outside. Now define the moment of inertia tensor,

$$I_{ij} \equiv \int \rho(\mathbf{r}) \left(\delta_{ij} r^2 - r_i r_j\right) d^3x$$

which depends only on the particular rigid body. This tensor is symmetric,

$$I_{ij} = I_{ji}$$

The torque equation may now be written as

$$\tau_i = \frac{d}{dt}\left(I_{ij}\omega_j\right)$$

We have therefore shown that the angluar momentum is

$$L_i = I_{ij}\omega_j$$

where equation of motion in an inertial frame is simply

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

In general, I_{ij} is not proportional to the identity, so that the angular momentum and the angular velocity are not parallel.

3.1 Rotating reference frame and the Euler equation

Next, suppose we look at the equation of motion in a rotating frame of reference. We must replace the time derivative,

$$\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times$$

and the equation of motion becomes

$$\boldsymbol{\tau} = \left(\frac{d\mathbf{L}}{dt}\right)_b + \boldsymbol{\omega}\times\mathbf{L}$$

This is the Euler equation.

In order to use the Euler equation, it is helpful work in a particular frame of reference. Given our rotating frame, any constant orthogonal transformation of the basis takes us to another equivalent rotating frame, but with a different orientation of the basis vectors. Furthermore, we know that any symmetric matrix may be diagonalized by an orthogonal transformation. Therefore, it is possible to rotate our basis to one in which I_{ij} is diagonal. In this basis, we have

$$[I]_{ij} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

The three eigenvalues, I_{11}, I_{22} and I_{33} are called the principal moments of inertia.

If we now write out the Euler equation in components using the principal moments, we have

$$\tau_i = \frac{d}{dt} I_{ij}\omega_j + \varepsilon_{ijk}\omega_j I_{km}\omega_m$$

so writing each component separately,

$$\begin{aligned}
\tau_1 &= \frac{d}{dt} I_{1j}\omega_j + \varepsilon_{1jk}\omega_j I_{km}\omega_m \\
&= \frac{d}{dt} I_{11}\omega_1 + \varepsilon_{123}\omega_2 I_{3m}\omega_m + \varepsilon_{132}\omega_3 I_{2m}\omega_m \\
&= I_{11}\frac{d\omega_1}{dt} + \varepsilon_{123}\omega_2 I_{33}\omega_3 + \varepsilon_{132}\omega_3 I_{22}\omega_2 \\
&= I_{11}\frac{d\omega_1}{dt} + \omega_2\omega_3 \left(I_{33} - I_{22}\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\tau_2 &= \frac{d}{dt} I_{22}\omega_2 + \varepsilon_{231}\omega_3 I_{11}\omega_1 + \varepsilon_{213}\omega_1 I_{33}\omega_3 \\
&= I_{22}\frac{d\omega_2}{dt} + \omega_3\omega_1 \left(I_{11} - I_{33}\right)
\end{aligned}$$

and

$$\begin{aligned}
\tau_3 &= \frac{d}{dt} I_{33}\omega_3 + \varepsilon_{312}\omega_1 I_{22}\omega_2 + \varepsilon_{321}\omega_2 I_{11}\omega_1 \\
&= I_{33}\frac{d\omega_3}{dt} + \omega_1\omega_2 \left(I_{22} - I_{11}\right)
\end{aligned}$$

Introducing the briefer (but potentially misleading) notation

$$\begin{aligned}
I_1 &= I_{11} \\
I_2 &= I_{22} \\
I_3 &= I_{33}
\end{aligned}$$

we have the Euler equations in the form0

$$\begin{aligned}
\tau_1 &= I_1\dot{\omega}_1 - \omega_2\omega_3 \left(I_2 - I_3\right) \\
\tau_2 &= I_2\dot{\omega}_2 - \omega_3\omega_1 \left(I_3 - I_1\right) \\
\tau_3 &= I_3\dot{\omega}_3 - \omega_1\omega_2 \left(I_1 - I_2\right)
\end{aligned}$$

3.2 Torque-free motion

When the torque vanishes, both the kinetic energy and the angular momentum are conserved. To find the kinetic energy, we write the action. There is no potential; in the inertial frame, the kinetic energy is the integral over the rigid body,

$$\begin{aligned}
T &= \frac{1}{2}\int \rho\left(\mathbf{r}\right)\mathbf{v}^2 d^3x \\
&= \frac{1}{2}\int \rho\left(\mathbf{r}\right)\left(\boldsymbol{\omega}\times\mathbf{r}\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}\right) d^3x \\
&= \frac{1}{2}\int \rho\left(\mathbf{r}\right)\left(\varepsilon_{imn}\omega_m r_n\right)\left(\varepsilon_{ijk}\omega_j r_k\right) d^3x
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\int \rho(\mathbf{r})\left(\varepsilon_{mni}\varepsilon_{jki}\right)\omega_m r_n \omega_j r_k d^3x \\
&= \frac{1}{2}\int \rho(\mathbf{r})\left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right)\omega_m r_n \omega_j r_k d^3x \\
&= \frac{1}{2}\omega_m\omega_j \int \rho(\mathbf{r})\left(\delta_{mj} r_n r_n - r_j r_m\right) d^3x \\
&= \frac{1}{2}\omega_m\omega_j I_{mj}
\end{aligned}$$

so we have

$$T = \frac{1}{2} I_{ij}\omega_i\omega_j$$

The action is therefore

$$S = \int \frac{1}{2} I_{ij}\omega_i\omega_j dt$$

where $\omega_i = \dot{\varphi} n_i$. Since there is no explicit time dependence, the energy

$$\begin{aligned}
E &= \frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi} - L \\
&= \left(I_{ij} n_i n_j \dot{\varphi}\right)\dot{\varphi} - \frac{1}{2} I_{ij}\omega_i\omega_j \\
&= \frac{1}{2} I_{ij}\omega_i\omega_j
\end{aligned}$$

is conserved. We also know that the total angular momentum is conserved,

$$L_i = I_{ij}\omega_j$$

Suppose, for concreteness, that $I_3 < I_2 < I_1$. The case when two of the principal moments are equal is simpler and will be examined separately. Then for torque-free motion, the Euler equations become

$$\begin{aligned}
I_1\dot{\omega}_1 &= \omega_2\omega_3\left(I_2 - I_3\right) \\
I_2\dot{\omega}_2 &= -\omega_3\omega_1\left(I_1 - I_3\right) \\
I_3\dot{\omega}_3 &= \omega_1\omega_2\left(I_1 - I_2\right)
\end{aligned}$$

where the differences on the right are non-negative. Add multiples of the first pair:

$$\begin{aligned}
I_1\left(I_1 - I_3\right)\omega_1\dot{\omega}_1 &= \omega_1\omega_2\omega_3\left(I_1 - I_3\right)\left(I_2 - I_3\right) \\
I_2\left(I_2 - I_3\right)\omega_2\dot{\omega}_2 &= -\omega_2\omega_3\omega_1\left(I_1 - I_3\right)\left(I_2 - I_3\right)
\end{aligned}$$

to find

$$\begin{aligned}
0 &= I_1\left(I_1 - I_3\right)\omega_1\dot{\omega}_1 + I_2\left(I_2 - I_3\right)\omega_2\dot{\omega}_2 \\
&= \frac{1}{2}\frac{d}{dt}\left(I_1\left(I_1 - I_3\right)\omega_1^2 + I_2\left(I_2 - I_3\right)\omega_2^2\right)
\end{aligned}$$

so with A constant, we have

$$I_1\left(I_1 - I_3\right)\omega_1^2 + I_2\left(I_2 - I_3\right)\omega_2^2 = A$$

Similarly, we find a relation between ω_3^2 and ω_2^2,

$$\begin{aligned}
I_2\left(I_1 - I_2\right)\omega_2\dot{\omega}_2 &= -\omega_3\omega_1\omega_2\left(I_1 - I_3\right)\left(I_1 - I_2\right) \\
I_3\left(I_1 - I_3\right)\omega_3\dot{\omega}_3 &= \omega_1\omega_2\omega_3\left(I_1 - I_2\right)\left(I_1 - I_3\right)
\end{aligned}$$

so

$$0 = \frac{1}{2}\frac{d}{dt}\left(I_2(I_1 - I_2)\omega_2^2 + I_3(I_1 - I_3)\omega_3^2\right)$$

Calling the second constant B, we solve for two of the components,

$$\begin{aligned} I_1\omega_1^2 &= \frac{1}{I_1 - I_3}\left[A - I_2(I_2 - I_3)\omega_2^2\right] \\ I_3\omega_3^2 &= \frac{1}{I_1 - I_3}\left[B - I_2(I_1 - I_2)\omega_2^2\right] \end{aligned}$$

Substituting into the energy,

$$\begin{aligned} 2E &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\ &= \frac{1}{I_1 - I_3}\left[A - I_2(I_2 - I_3)\omega_2^2\right] + I_2\omega_2^2 + \frac{1}{I_1 - I_3}\left[B - I_2(I_1 - I_2)\omega_2^2\right] \\ &= \frac{1}{I_1 - I_3}\left(A + B - I_2(I_2 - I_3)\omega_2^2 + I_2(I_1 - I_3)\omega_2^2 - I_2(I_1 - I_2)\omega_2^2\right) \\ &= \frac{1}{I_1 - I_3}\left(A + B + (-I_2I_2 + I_2I_3 + I_1I_2 - I_2I_3 - I_1I_2 + I_2I_2)\omega_2^2\right) \\ &= \frac{1}{I_1 - I_3}(A + B) \end{aligned}$$

so the sum of the constants is related to the energy,

$$A + B = 2(I_1 - I_3)E$$

To find the remaining component, we solve for ω_1 and ω_3,

$$\begin{aligned} \omega_1 &= \sqrt{\frac{1}{I_1(I_1 - I_3)}\left(A - I_2(I_2 - I_3)\omega_2^2\right)} \\ \omega_3 &= \sqrt{\frac{1}{I_3(I_1 - I_3)}\left(B - I_2(I_1 - I_2)\omega_2^2\right)} \end{aligned}$$

and substitute into the differentital equation for ω_2,

$$I_2\dot{\omega}_2 = -\omega_3\omega_1(I_1 - I_3)$$

Integrating gives an expression which can be written in terms of elliptic integrals,

$$-\sqrt{\frac{I_1I_2I_3}{AB}}t = \int \frac{d\omega_2}{\sqrt{\left(1 - \frac{I_2(I_2-I_3)}{A}\omega_2^2\right)\left(1 - \frac{I_2(I_1-I_2)}{B}\omega_2^2\right)}}$$

Rescale ω_2, letting

$$\chi = \sqrt{\frac{I_2(I_2 - I_3)}{A}}\omega_2$$

Then

$$-\sqrt{\frac{I_2(I_2 - I_3)}{A}}\sqrt{\frac{I_1I_2I_3}{AB}}t = \int \frac{d\chi}{\sqrt{(1 - \chi^2)(1 - k^2\chi^2)}}$$

where

$$k^2 = \frac{A(I_1 - I_2)}{B(I_2 - I_3)}$$

The right side is Jacobi's form of the elliptic integral of the first kind,

$$F(x,k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

so we have

$$-\sqrt{\frac{I_2(I_2-I_3)}{A}}\sqrt{\frac{I_1I_2I_3}{AB}}t = F\left(\sqrt{\frac{I_2(I_2-I_3)}{A}}\omega_2, \frac{A(I_1-I_2)}{B(I_2-I_3)}\right)$$

We show below that for a symmetric body, the torque-free solution is much easier to understand.

3.3 Torque-free motion of a symmetric rigid body

Now consider the case when two of the moments of inertia are equal. This happens when th rigid body is rotationally symmetric around one axis. Let the z-axis be the axis of symmetry. Then $I_1 = I_2$, and the torque-free Euler equations become

$$\begin{aligned} 0 &= I_1\dot{\omega}_1 - \omega_2\omega_3(I_1-I_3) \\ 0 &= I_1\dot{\omega}_2 + \omega_1\omega_3(I_1-I_3) \\ 0 &= I_3\dot{\omega}_3 \end{aligned}$$

The final equation shows that ω_3 is constant. Defining the constant frequency

$$\Omega \equiv \omega_3\left(\frac{I_1-I_3}{I_1}\right)$$

the remaining two equations are

$$\begin{aligned} \dot{\omega}_1 &= \Omega\omega_2 \\ \dot{\omega}_2 &= -\Omega\omega_1 \end{aligned}$$

We decouple these by differentiating the first and substituting the second,

$$\begin{aligned} \ddot{\omega}_1 &= \Omega\dot{\omega}_2 \\ &= -\Omega^2\omega_1 \end{aligned}$$

and similarly, by differentiating the second and substituting the first. This results in the pair

$$\begin{aligned} \ddot{\omega}_1 + \Omega^2\omega_1 &= 0 \\ \ddot{\omega}_2 + \Omega^2\omega_2 &= 0 \end{aligned}$$

with the immediate solution

$$\omega_1 = A\cos\Omega t + B\sin\Omega t$$

for ω_1 and, returning to the original equation $\dot{\omega}_1 = \Omega\omega_2$,

$$\omega_2 = -A\sin\Omega t + B\cos\Omega t$$

Notice that

$$\omega_1^2 + \omega_2^2 = A^2 + B^2$$

so the x and y components of the angular velocity together form a constant length vector that precesses around the z axis. If the angular velocity is dominated by ω_3, the remaining components give the object a "wobble" – it spins slightly off its symmetry axis, precessing. On the other hand, if ω_3 is small, the motion is a "tumble" – end over end rotation *of* its symmetry axis.

Remember that this analysis takes place in a frame of reference rotating with angular velocity $\boldsymbol{\omega}$. If all of the motion were about the z-axis, the object would be at rest in the rotating frame. The fact that we get time dependence of our solution for $\boldsymbol{\omega}$ means that even in a frame rotating with the body, the body precesses. If we transform back to the inertial frame, it is also spinning.

4 Lagrangian approach to rigid bodies

To study symmetric rigid bodies with one point fixed and gravity acting – tops – we begin afresh and write an action for the problem. In order to do this, we require some set of coordinates. These are taken to be the Euler angles. There are actually many ways to define a useful set of three angles; we follow the definition used in Goldstein, Section 4.4.

Our goal is to related a fixed inertial system, (x', y', z') to a set of Cartesian axes fixed in the top, (x, y, z). The relationship is defined by concatenating three rotations:

1. Rotate about the z-axis through an angle φ, giving intermetiate coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. Call this coordinate transformation matrix D.

2. Rotate about the ξ_1 axis by an angle θ, giving coordinates $\boldsymbol{\xi}'$. Call this coordinate transformation matrix C.

3. Rotate about ξ_3' by an angle ψ to the final $\mathbf{x}$ coordinates. Call this coordinate transformation matrix B.

We may think of (θ, φ) as the direction of the symmetry axis of the top, with ψ giving its angle of rotation about that axis. It is easy to construct the full transformation between $\mathbf{x}'$ and $\mathbf{x}$ because each of these transformations, D, C, B is a simple 2-dim rotation. The full transformation, A, is therefore just the product

$$\begin{aligned} \mathbf{x} &= A\mathbf{x}' \\ &= BCD\mathbf{x}' \end{aligned}$$

of the three, applying D first, then C, then B, where

$$\begin{aligned} D &= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \\ B &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Multiplying this out, we have

$$\begin{aligned} A &= BCD \\ &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\cos\theta\sin\varphi & \cos\theta\cos\varphi & \sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \end{aligned}$$

The inverse transformation is just the transpose, $A^t = A^{-1}$.

Now denote the angular velocity vector of the rigid body as $\boldsymbol{\omega}'$ with respect to the inertial frame or reference, and let this velocity be the time derivative of the Euler angles, , $\left(\dot{\varphi}, \dot{\theta}, \dot{\psi}\right)$. We can write this a vector relationship using the intermetiate coordinates,

$$\boldsymbol{\omega} = \dot{\varphi}\hat{\mathbf{z}}' + \dot{\theta}\hat{\boldsymbol{\xi}}_1 + \dot{\psi}\hat{\mathbf{z}}$$

Using A, B, C and D we can find these components with respect to the body frame. For the first term, $\dot{\varphi}\hat{\mathbf{z}}'$ we write

$$\hat{\mathbf{z}}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and write this in terms of the body basis as

$$\begin{aligned} A\hat{\mathbf{z}}' &= \begin{pmatrix} \sin\psi\sin\theta \\ \cos\psi\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\mathbf{x}}\sin\psi\sin\theta + \hat{\mathbf{y}}\cos\psi\sin\theta + \hat{\mathbf{z}}\cos\theta \\ \dot{\varphi}\hat{\mathbf{z}}' &= \hat{\mathbf{x}}\dot{\varphi}\sin\psi\sin\theta + \hat{\mathbf{y}}\dot{\varphi}\cos\psi\sin\theta + \hat{\mathbf{z}}\dot{\varphi}\cos\theta \end{aligned}$$

For the next term, $\dot{\theta}\hat{\boldsymbol{\xi}}_1$, we only need the final rotation to get to the body system, since $\hat{\boldsymbol{\xi}}_1 = \hat{\boldsymbol{\xi}}_1'$. Therefore, we compute

$$\begin{aligned} B\dot{\theta}\hat{\boldsymbol{\xi}}_1 &= \dot{\theta}B\hat{\boldsymbol{\xi}}_1' \\ &= \dot{\theta}\begin{pmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{pmatrix} = \hat{\mathbf{x}}\dot{\theta}\cos\psi - \hat{\mathbf{y}}\dot{\theta}\sin\psi \end{aligned}$$

Finally, $\dot{\psi}\hat{\mathbf{z}}$ is already in the body frame. Adding these, we have

$$\begin{aligned} \boldsymbol{\omega} &= \dot{\varphi}\hat{\mathbf{z}}' + \dot{\theta}\hat{\boldsymbol{\xi}}_1 + \dot{\psi}\hat{\mathbf{z}} \\ &= \left(\dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\right)\hat{\mathbf{x}} + \left(\dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi\right)\hat{\mathbf{y}} + \left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\hat{\mathbf{z}} \end{aligned}$$

4.1 Action functional for rigid body motion

We are now in a position to write the Lagrangian and action for a rigid body. In terms of Euler coordinates, the kinetic energy is

$$T = \frac{1}{2}I_{ij}\omega_i\omega_j$$

Substituting for the angular velocity in the principal axis frame this becomes

$$T = \frac{1}{2}I_1\left(\dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2$$

We may also have the kinetic energy of the center of mass.

For a slowly changing force field, we may write the potential as a function of the center of mass only, but if there is a gradient or the forces are applied at specific points of the rigid body, there may be torques as well. If the body is in a gravitational field, the potential is found by integrating

$$dV = -dm\mathbf{g}(\mathbf{r}) \cdot d\mathbf{x}$$

where $\mathbf{g}(\mathbf{x})$ is the local gravitational acceleration. For a uniform gravitational field, $\mathbf{g} = -g\mathbf{k}$ is constant so $\int \mathbf{g}(\mathbf{r}) \cdot d\mathbf{x} = \mathbf{g} \cdot \mathbf{x}$

$$\begin{aligned} dV &= -\left(\rho d^3x\right)\mathbf{g}\cdot\mathbf{x} \\ V &= -\mathbf{g}\cdot\int \rho\mathbf{x}d^3x \\ &= -M\mathbf{g}\cdot\mathbf{R} \end{aligned}$$

since the center of mass is defined as

$$\mathbf{R} = \frac{1}{M}\int \rho \mathbf{x} d^3x$$

We now apply these considerations to the case of a rigid body symmetric about one axis, with one point fixed: tops.

4.2 Symmetric body with torque: tops

Now suppose the rotationally symmetric body rests on one point of the symmetry axis, like a top spinning on a tabletop. We take this point as fixed. Then, unless the top is perfectly vertical, there is a torque acting, produced by gravity acting at the center of mass. If the center of mass is a distance l above the fixed tip, then the potential is

$$V = Mgl\cos\theta$$

Taking the z-axis as the symmetry axis, we have $I_1 = I_2$, and the first two terms of the kinetic energy simplify considerably. The cross terms cancel and the sums of squares combine to give

$$\frac{1}{2}I_1\left(\left(\dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\right)^2 + \left(\dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi\right)^2\right) = \frac{1}{2}I_1\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right)$$

Then the action becomes

$$S = \int dt\left(\frac{1}{2}I_1\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 - Mgl\cos\theta\right)$$

We first look for conserved quantities. Two angles, φ and ψ, are cyclic, so their conjugate momenta are conserved:

$$\begin{aligned} p_\varphi &= \frac{\partial L}{\partial\dot{\varphi}} \\ &= I_1\dot{\varphi}\sin^2\theta + I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\cos\theta \\ &= \dot{\varphi}\left(I_1\sin^2\theta + I_3\cos^2\theta\right) + I_3\dot{\psi}\cos\theta \end{aligned}$$

and

$$\begin{aligned} p_\psi &= \frac{\partial L}{\partial\dot{\psi}} \\ &= I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right) \\ &= I_3\omega_3 \end{aligned}$$

The energy provides a third constant of the motion since $\frac{\partial L}{\partial t} = 0$. Since the Lagrangian is quadratic in the velocities, we have $E = T + V$,

$$\begin{aligned} E &= \frac{1}{2}I_1\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 + Mgl\cos\theta \\ &= \frac{1}{2}I_1\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{p_\psi^2}{2I_3} + Mgl\cos\theta \end{aligned}$$

To solve, we first eliminate $\dot{\psi}$,

$$\dot{\psi} = \frac{p_\psi}{I_3} - \dot{\varphi}\cos\theta$$

then substitute this into p_φ,

$$\begin{aligned} p_\varphi &= \dot{\varphi}\left(I_1 \sin^2\theta + I_3\cos^2\theta\right) + I_3\left(\frac{p_\psi}{I_3} - \dot{\varphi}\cos\theta\right)\cos\theta \\ &= \dot{\varphi}\left(I_1 \sin^2\theta + I_3\cos^2\theta\right) + p_\psi\cos\theta - I_3\dot{\varphi}\cos^2\theta \\ &= \dot{\varphi} I_1 \sin^2\theta + p_\psi\cos\theta \end{aligned}$$

so that we may also solve for $\dot{\varphi}$. This gives

$$\dot{\varphi} = \frac{p_\varphi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

Finally, we use the energy to express θ as an integral,

$$\begin{aligned} E &= \frac{1}{2}I_1\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{p_\psi^2}{2I_3} + Mgl\cos\theta \\ &= \frac{1}{2}I_1\left(\left(\frac{p_\varphi - p_\psi\cos\theta}{I_1\sin^2\theta}\right)^2\sin^2\theta + \dot{\theta}^2\right) + \frac{p_\psi^2}{2I_3} + Mgl\cos\theta \\ &= \frac{1}{2}I_1\dot{\theta}^2 + \frac{\left(p_\varphi - p_\psi\cos\theta\right)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mgl\cos\theta \end{aligned}$$

We may drop the constant term, $\frac{p_\psi^2}{2I_3}$. Then, solving for $\dot{\theta}$ to integrate,

$$\begin{aligned} t &= \int \frac{d\theta}{\sqrt{\frac{2E}{I_1} - \frac{(p_\varphi - p_\psi\cos\theta)^2}{I_1^2\sin^2\theta} - \frac{2Mgl}{I_1}\cos\theta}} \\ &= \int \frac{\sin\theta d\theta}{\sqrt{\frac{2E}{I_1}\sin^2\theta - \frac{1}{I_1^2}\left(p_\varphi - p_\psi\cos\theta\right)^2 - \frac{2Mgl}{I_1}\sin^2\theta\cos\theta}} \end{aligned}$$

or, setting $x = \cos\theta$,

$$t = -\int \frac{dx}{\sqrt{\frac{2E}{I_1}\left(1 - x^2\right) - \frac{1}{I_1^2}\left(p_\varphi - p_\psi x\right)^2 - \frac{2Mgl}{I_1}\left(x - x^3\right)}}$$

The cubic in under the root makes this difficult, but it can be expressed in terms of elliptic integrals or numerically integrated. The resulting $\theta(t)$ then allows us to integrate to find $\varphi(t)$ and $\psi(t)$.

A simpler way to approach the qualitative behavior is to view the energy as that of a 1-dimensional problem with an effective potential

$$V_{eff} = \frac{\left(p_\varphi - p_\psi\cos\theta\right)^2}{2I_1\sin^2\theta} + Mgl\cos\theta$$

where we again drop the irrelevant constant, $\frac{p_\psi^2}{2I_3}$.

To explore the motion in this potential, again set $x = \cos\theta$. Then

$$V_{eff} = \frac{\left(p_\varphi - p_\psi x\right)^2}{2I_1\left(1 - x^2\right)} + Mglx$$

This has extrema when

$$0 = \frac{dV_{eff}}{dx}$$

$$
\begin{aligned}
&= \frac{-2p_\psi (p_\varphi - p_\psi x)}{2I_1 (1-x^2)} - \frac{(p_\varphi - p_\psi x)^2}{2I_1 (1-x^2)^2}(-2x) + Mgl \\
&= \frac{1}{2I_1 (1-x^2)^2}\left(-p_\psi 2 (p_\varphi - p_\psi x)\left(1-x^2\right) + 2x (p_\varphi - p_\psi x)^2 + 2I_1 Mgl \left(1-x^2\right)^2\right) \\
0 &= -2p_\psi p_\varphi \left(1-x^2\right) + 2p_\psi^2 x \left(1-x^2\right) + 2p_\varphi^2 x - 4p_\varphi p_\psi x^2 + 2p_\psi^2 x^3 + 2I_1 Mgl \left(1 - 2x^2 + x^4\right) \\
0 &= (2I_1 Mgl - 2p_\psi p_\varphi) + \left(2p_\varphi^2 + 2p_\psi^2\right) x + (2p_\psi p_\varphi - 4p_\varphi p_\psi - 4I_1 Mgl) x^2 + \left(2p_\psi^2 - 2p_\psi^2\right) x^3 + 2I_1 Mgl x^4 \\
0 &= (I_1 Mgl - p_\psi p_\varphi) + \left(p_\varphi^2 + p_\psi^2\right) x - (p_\varphi p_\psi + 2I_1 Mgl) x^2 + (I_1 Mgl) x^4 \\
0 &= I_1 Mgl \left(1-x^2\right)^2 - p_\psi p_\varphi \left(1+x^2\right) + \left(p_\varphi^2 + p_\psi^2\right) x
\end{aligned}
$$

For x near 1, the first term may be neglected and we have approximately

$$
\begin{aligned}
0 &= p_\psi p_\varphi - \left(p_\varphi^2 + p_\psi^2\right) x + p_\psi p_\varphi x^2 \\
x &= \frac{1}{2p_\psi p_\varphi}\left(\left(p_\varphi^2 + p_\psi^2\right) \pm \sqrt{\left(p_\varphi^2 + p_\psi^2\right)^2 - 4p_\psi^2 p_\varphi^2}\right) \\
x &= \frac{1}{2p_\psi p_\varphi}\left(p_\varphi^2 + p_\psi^2 \pm \left(p_\varphi^2 - p_\psi^2\right)\right) \\
x &= \frac{p_\varphi}{p_\psi}, \frac{p_\psi}{p_\varphi}
\end{aligned}
$$

In this case the top precesses in a nearly vertical position at a speed near its rate of spin.

Now consider small x, so we neglect the x^4 term. Then

$$
\begin{aligned}
0 &= (p_\psi p_\varphi + 2I_1 Mgl) x^2 - \left(p_\varphi^2 + p_\psi^2\right) x - (I_1 Mgl - p_\psi p_\varphi) \\
x &= \frac{1}{2(p_\psi p_\varphi + 2I_1 Mgl)}\left(\left(p_\varphi^2 + p_\psi^2\right) \pm \sqrt{\left(p_\psi^2 + p_\varphi^2\right)^2 + 4(p_\psi p_\varphi + 2I_1 Mgl)(I_1 Mgl - p_\psi p_\varphi)}\right) \\
x &= \frac{1}{2(p_\psi p_\varphi + 2I_1 Mgl)}\left(\left(p_\varphi^2 + p_\psi^2\right) \pm \sqrt{\left(p_\psi^2 - p_\varphi^2\right)^2 - 4I_1 Mgl p_\psi p_\varphi + 8I_1^2 M^2 g^2 l^2}\right)
\end{aligned}
$$

This has real solutions as long as

$$
\left(p_\psi^2 - p_\varphi^2\right)^2 + 8I_1^2 M^2 g^2 l^2 \geq 4I_1 Mgl p_\psi p_\varphi
$$

When the spin is fast, we have $p_\psi \gg p_\varphi$ and may approximate

$$
p_\psi^4 + 8I_1^2 M^2 g^2 l^2 \geq 4I_1 Mgl p_\psi^2 \frac{p_\varphi}{p_\psi}
$$

But then

$$
\begin{aligned}
0 &\leq \left(p_\psi^2 - 2I_1 Mgl\right)^2 \\
&= p_\psi^4 - 4I_1 Mgl p_\psi^2 + 4I_1^2 M^2 g^2 l^2 \\
p_\psi^4 + 4I_1^2 M^2 g^2 l^2 &\geq 4I_1 Mgl p_\psi
\end{aligned}
$$

and since $\frac{p_\varphi}{p_\psi} < 1$, there is always a solution. When the energy equals the potential at such a minimum, the top will precess in a circle at a fixed angle θ. We may the consider perturbations around this solution. Various solutions are depicted in Figure 5.9 of Goldstein, depending on the relative frequencies in the θ and φ oscillations.

4.2.1 Slowly precessing top

Consider the case when we have $\dot{\psi} \gg \dot{\theta} \gg \dot{\varphi}$. Then we may make the following approximations:

$$\begin{aligned} \dot{\psi} &= \frac{p_\psi}{I_3} - \dot{\varphi}\cos\theta \\ &\approx \frac{p_\psi}{I_3} \end{aligned}$$

and

$$\dot{\varphi} = \frac{p_\varphi - p_\psi \cos\theta}{I_1 \sin^2\theta} \ll \dot{\psi}$$

while the energy is approximately

$$\begin{aligned} E &= \frac{1}{2} I_1 \left(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{p_\psi^2}{2I_3} + Mgl\cos\theta \\ E' = \left(E - \frac{p_\psi^2}{2I_3} \right) &\approx \frac{1}{2} I_1 \dot{\theta}^2 + Mgl\cos\theta \end{aligned}$$

Then, solving for $\dot{\theta}$ to integrate, we find an elliptic integral:

$$\begin{aligned} t &= \int \frac{d\theta}{\sqrt{\frac{2E'}{I_1} - \frac{2Mgl}{I_1}\cos\theta}} \\ &= \sqrt{\frac{2I_1}{E' - Mgl}} F\left(\frac{\theta}{2}; -\frac{4Mgl}{2E' - 2Mgl} \right) \end{aligned}$$

4.3 Gyroscopes

Gyroscopes are typically mounted on freely turning frames so that there is no external torque. In this case, the potential vanishes and we have the simpler system

$$\begin{aligned} \dot{\psi} &= \frac{p_\psi}{I_3} - \dot{\varphi}\cos\theta \\ \dot{\varphi} &= \frac{p_\varphi - p_\psi\cos\theta}{I_1 \sin^2\theta} \\ E &= \frac{1}{2} I_1 \left(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{p_\psi^2}{2I_3} \end{aligned}$$

with effective potential

$$V_{eff} = \frac{(p_\varphi - p_\psi\cos\theta)^2}{2I_1 \sin^2\theta}$$

The extrema at

$$x = \min\left(\frac{p_\varphi}{p_\psi}, \frac{p_\psi}{p_\varphi} \right)$$

where the minimum selects for the value which gives $x \leq 1$. There is therefore exactly one solution

$$t = \int \frac{d\theta}{\sqrt{\frac{2E}{I_1} - \frac{(p_\varphi - p_\psi\cos\theta)^2}{I_1^2 \sin^2\theta}}}$$

$$
\begin{aligned}
&= \int \frac{\sin\theta d\theta}{\sqrt{\frac{2E}{I_1}\sin^2\theta - \frac{1}{I_1^2}\left(p_\varphi - p_\psi\cos\theta\right)^2}} \\
&= \int \frac{I_1 dx}{\sqrt{\left(2I_1E - p_\varphi^2\right) + 2p_\varphi p_\psi x - \left(2I_1E + p_\psi^2\right)x^2}}
\end{aligned}
$$

This time the root is quadratic, and we may complete the square,

$$
\left(2I_1E - p_\varphi^2\right) + 2p_\varphi p_\psi x - \left(2I_1E + p_\psi^2\right)x^2 = -\left(\sqrt{2I_1E + p_\psi^2}\,x - \frac{p_\varphi p_\psi}{\sqrt{2I_1E + p_\psi^2}}\right)^2 + \frac{p_\varphi^2 p_\psi^2}{2I_1E + p_\psi^2} + 2I_1E - p_\varphi^2
$$

Setting

$$
\begin{aligned}
\xi &= \sqrt{2I_1E + p_\psi^2}\,x - \frac{p_\varphi p_\psi}{\sqrt{2I_1E + p_\psi^2}} \\
dx &= \frac{d\xi}{\sqrt{2I_1E + p_\psi^2}} \\
A^2 &= \frac{p_\varphi^2 p_\psi^2}{2I_1E + p_\psi^2} + 2I_1E - p_\varphi^2 \\
\Omega &= \frac{1}{I_1}\sqrt{2I_1E + p_\psi^2}
\end{aligned}
$$

we have

$$
t = \frac{I_1}{\sqrt{2I_1E + p_\psi^2}} \int \frac{d\xi}{\sqrt{A^2 - \xi^2}}
$$

so we set

$$
\xi = A\sin\alpha
$$

and the integral is

$$
\begin{aligned}
t &= \frac{I_1}{\sqrt{2I_1E + p_\psi^2}} \sin^{-1}\frac{\xi}{A} \\
\sqrt{2I_1E + p_\psi^2}\cos\theta - \frac{p_\varphi p_\psi}{\sqrt{2I_1E + p_\psi^2}} &= A\sin\Omega t \\
\cos\theta &= \frac{p_\varphi p_\psi}{2I_1E + p_\psi^2} + \frac{A}{\sqrt{2I_1E + p_\psi^2}}\sin\Omega t \\
&= \cos\theta_0 + b\sin\Omega t
\end{aligned}
$$

This displays nutation clearly: the tip angle of the gyroscope oscillates up and down around the angle θ_0 with period Ω. From $\Omega = \frac{1}{I_1}\sqrt{2I_1E + p_\psi^2}$ we see that Ω may have any magnitude, depending on the size of I_1. Then, from

$$
\dot\varphi = \frac{p_\varphi - p_\psi\cos\theta}{I_1\sin^2\theta}
$$

we see that the value of p_φ makes $\dot\varphi$ independent of Ω. This means that the rate of precession and the rate of nutation are independent of one another.