Connections to quantum mechanics

December 2, 2014

1 Quantum Mechanics and the Hamilton-Jacobi equation

The Hamiltonian-Jacobi equation provides the most direct link between classical and quantum mechanics. There is considerable similarity between the Hamilton-Jacobi equation and the Schrödinger equation:

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial t} &= -H(x_i, \frac{\partial \mathcal{S}}{\partial x_i}, t) \\ i\hbar \frac{\partial \psi}{\partial t} &= H(\hat{x}_i, \hat{p}_i, t) \end{aligned}$$

We make the relationship precise as follows.

Suppose the Hamiltonian in each case is that of a single particle in a potential:

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})$$

Write the quantum wave function as

$$\psi = A e^{\frac{i}{\hbar}\varphi}$$

The Schrödinger equation becomes

$$\begin{aligned} i\hbar \frac{\partial \left(A e^{\frac{i}{\hbar}\varphi} \right)}{\partial t} &= -\frac{\hbar^2}{2m} \bigtriangledown^2 \left(A e^{\frac{i}{\hbar}\varphi} \right) + V \left(A e^{\frac{i}{\hbar}\varphi} \right) \\ i\hbar \frac{\partial A}{\partial t} e^{\frac{i}{\hbar}\varphi} - A e^{\frac{i}{\hbar}\varphi} \frac{\partial \varphi}{\partial t} &= -\frac{\hbar^2}{2m} \bigtriangledown \cdot \left(e^{\frac{i}{\hbar}\varphi} \bigtriangledown A + \frac{i}{\hbar} A e^{\frac{i}{\hbar}\varphi} \bigtriangledown \varphi \right) + V A e^{\frac{i}{\hbar}\varphi} \\ &= -\frac{\hbar^2}{2m} e^{\frac{i}{\hbar}\varphi} \left(\frac{i}{\hbar} \bigtriangledown \varphi \bigtriangledown A + \bigtriangledown^2 A \right) \\ &\quad -\frac{\hbar^2}{2m} e^{\frac{i}{\hbar}\varphi} \left(\frac{i}{\hbar} \bigtriangledown A \cdot \bigtriangledown \varphi + \frac{i}{\hbar} A \bigtriangledown^2 \varphi \right) \\ &\quad -\frac{\hbar^2}{2m} \left(\frac{i}{\hbar} \right)^2 e^{\frac{i}{\hbar}\varphi} \left(A \bigtriangledown \varphi \cdot \bigtriangledown \varphi \right) \\ &\quad + V A e^{\frac{i}{\hbar}\varphi} \end{aligned}$$

Then cancelling the exponential,

$$\begin{aligned} i\hbar \frac{\partial A}{\partial t} - A \frac{\partial \varphi}{\partial t} &= -\frac{i\hbar}{2m} \bigtriangledown \varphi \bigtriangledown A - \frac{\hbar^2}{2m} \bigtriangledown^2 A \\ &\quad -\frac{i\hbar}{2m} \bigtriangledown A \cdot \bigtriangledown \varphi - \frac{i\hbar}{2m} A \bigtriangledown^2 \varphi \\ &\quad + \frac{1}{2m} \left(A \bigtriangledown \varphi \cdot \bigtriangledown \varphi \right) + V A \end{aligned}$$

Collecting by powers of $\hbar$,

$$\begin{aligned} O\left(\hbar^0\right) &: \quad -\frac{\partial \varphi}{\partial t} = \frac{1}{2m} \bigtriangledown \varphi \cdot \bigtriangledown \varphi + V \\ O\left(\hbar^1\right) &: \quad \frac{1}{A}\frac{\partial A}{\partial t} = -\frac{1}{2m}\left(\frac{2}{A} \bigtriangledown A \cdot \bigtriangledown \varphi + \bigtriangledown^2 \varphi\right) \\ O\left(\hbar^2\right) &: \quad 0 = -\frac{\hbar^2}{2m} \bigtriangledown^2 A \end{aligned}$$

The zeroth order terms is the Hamilton-Jacobi equation, with $\varphi = \mathcal{S}$:

$$\begin{aligned} -\frac{\partial \mathcal{S}}{\partial t} &= \frac{1}{2m} \bigtriangledown \mathcal{S} \cdot \bigtriangledown \mathcal{S} + V \\ &= \frac{1}{2m}\mathbf{p}^2 + V(x) \end{aligned}$$

where $p = \bigtriangledown \mathcal{S}$. Therefore, the Hamilton-Jacobi equation is the $\hbar \to 0$ limit of the Schrödinger equation.

$$H\psi = \frac{p^2}{2m}\psi + V\left(x\right)\psi$$