Problems 4c

November 7, 2012

Practice with index notation

Borrowed from Jackson, *Classical Electrodynamics*

1. Prove the following identities using index notation.

$$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}) \\ (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c}) \end{aligned}$$

2. The grad operator, ∇, is treated like a vector with components $\nabla_i = \frac{\partial}{\partial x_i}$, but it is also an operator. The thing to remember is that it always obeys the product rule. For example, for a function, f, and a vector, $\mathbf{a}$,

$$\begin{aligned} \nabla\cdot(f\mathbf{a}) &= \nabla_i\,(f a_i) \\ &= (\nabla_i f)\,a_i + f\nabla_i a_i \\ &= (\nabla f)\cdot\mathbf{a} + f\nabla\cdot\mathbf{a} \end{aligned}$$

   Prove the following two identities. Both of these require results involving symmetry:

$$\begin{aligned} \nabla\times\nabla f &= 0 \\ \nabla\cdot(\nabla\times\mathbf{a}) &= 0 \end{aligned}$$

3. Prove the following identities:

$$\begin{aligned} \nabla\times(f\mathbf{a}) &= (\nabla f)\times\mathbf{a} + f\nabla\times\mathbf{a} \\ \nabla\times(\nabla\times\mathbf{a}) &= \nabla\,(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a} \\ \nabla\,(\mathbf{a}\cdot\mathbf{b}) &= (\mathbf{a}\cdot\nabla)\,\mathbf{b} + (\mathbf{b}\cdot\nabla)\,\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}) \\ \nabla\cdot(\mathbf{a}\times\mathbf{b}) &= \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}) \\ \nabla\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}\,(\nabla\cdot\mathbf{b}) - \mathbf{b}\,(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\,\mathbf{a} - (\mathbf{a}\cdot\nabla)\,\mathbf{b} \end{aligned}$$