Problems 3b: Keplerian orbits

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1 Conic Sections

A general plane is described by

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$$

where the unit vector $\mathbf{n}$ gives the normal and the plane passes through the point $\mathbf{a}$. A cone, symmetric around the z axis, making an angle β with the z axis is described by

$$x^2 + y^2 = \alpha z$$

where

$$\cos\beta = \frac{z}{\sqrt{x^2 + y^2}} = \frac{z}{\rho}$$

Let $\mathbf{a} = a\mathbf{k}$, so the plane always passes through the point $(0, 0, a)$, and therefore intersects the cone for positive z. Show that the intersection of these gives the following curves:

- A *circle* when $\mathbf{n} = \mathbf{k}$
- An *ellipse*, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $\mathbf{n}$ points inside the cone
- A parabola when $\mathbf{n}$ is somewhere perpendicular to the side of the cone
- An hyperbola when $\mathbf{n}$ lies outside the cone

2 Moving between circular orbits

A satellite moves in a circular orbit around Earth at a radius r_1, and we wish to move it to a higher circular orbit at r_2. This is to be accomplished with two short bursts of the engine, giving energy boosts of E_1 and E_2. Assume E_1, E_2 are small enough that the satellite cannot escape a bound orbit.

1. Show that the first burst always produces an elliptical orbit with it's perigee (closest approach) at the point of the burst.
2. Determine the minimum E_1 required to change the orbit to r_2. Find where the second burst should be fired to minimize E_2, and find the appropriate value of E_2.

3 Kepler's laws

We have shown that orbits for the Newtonian 2-body problem are ellipses, and this is the first of Kepler's laws. Kepler's second and third laws are

- The orbiting body sweeps out equal areas in equal times
- The period and semimajor axis of the orbit are related by $T^2 = \frac{4\pi^2}{G(M+m)}a^3$.

Prove these two laws from our solution for the bound state motion. Hint: Use the second law to prove the third.

4 Geosynchronous orbits

A geosynchronous orbit is a circular orbit above the equator at an altitude that allows the satellite to stay above a fixed point on the surface of Earth.

1. Find the radius of a geosynchronous orbit

2. Consider a nearly geosynchronous orbit, where the angle of the orbit is tipped slightly from the equatorial plane. Find the curve on the surface of Earth which lies directly below the satellite as it completes a full orbit.