Problem Set I.a

August 28, 2014

1. a. Let a linear force on a particle of mass, m, always lie in the x-direction so that

$$\mathbf{F} = -kx\hat{\mathbf{i}}$$

Allowing the motion to be 2-dimensional, find the motion, $\mathbf{x}(t)$, with the initial conditions ($t_0 = 0$):

$$\begin{aligned} \mathbf{x}(0) &= x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} \\ \mathbf{v}(0) &= v_0\hat{\mathbf{j}} \end{aligned}$$

b. Let the initial conditions be

$$\begin{aligned} \mathbf{x}(0) &= x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} \\ \mathbf{v}(0) &= v_{0x}\hat{\mathbf{i}} + v_{0y}\hat{\mathbf{j}} \end{aligned}$$

Prove that the solutions are equivalent up to a different choice of the initial time t_0 and a specific releationship between v_0 and (v_{0x}, v_{0y}).

2. Two dimensional oscillator

Now consider a radial Hooke's law force in 2-dimensions,

$$\mathbf{F} = -kr\hat{\mathbf{r}}$$

where the force is along the radial unit vector $\hat{\mathbf{r}}$ and depends on the distance from the origin, r, where

$$\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{i}}\cos\varphi + \hat{\mathbf{j}}\sin\varphi \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

and therefore

$$\begin{aligned} \mathbf{r} &= r\hat{\mathbf{r}} \\ &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \end{aligned}$$

Let the initial position and velocity (at $t_0 = 0$) be

$$\begin{aligned} \mathbf{x}(0) &= x_0\hat{\mathbf{i}} \\ \mathbf{v}(0) &= v_0\hat{\mathbf{j}} \end{aligned}$$

Find the motion, $\mathbf{x}(t)$.

3. Solve for the motion of a block of mass, m, down an inclined plane of angle θ, which is connected by a rope to a rolling cart of mass M. Neglect the moment of inertia of the wheels of the cart and the pulley. Assume friction between the block and the inclined plane of

$$\mathbf{f} = -\mu\mathbf{N}$$

where $\mathbf{N}$ is the normal force exerted by the plane on the block.

4. We have proved that the angular momentum,

$$\mathbf{L} = (\mathbf{r}(t) - \mathbf{R}) \times \mathbf{p}(t)$$

of a single particle about a fixed position, $\mathbf{R}$, is related to the torque,

$$\mathbf{N} = (\mathbf{r}(t) - \mathbf{R}) \times \mathbf{F}(t)$$

by the equation

$$\mathbf{N} = \frac{d\mathbf{L}}{dt}$$

Now consider an isolated system of particles with positions $\mathbf{r}_i$ and momenta $\mathbf{p}_i$ where the force *on* the i^{th} particle *by* the j^{th} particle is $\mathbf{F}_{ij}$. Assume Newton's 3^{rd} Law holds, so that

$$\begin{aligned} \mathbf{F}_{ji} &= -\mathbf{F}_{ij} \\ \mathbf{F}_{ii} &= 0 \end{aligned}$$

Assume that the force between any two of the particles is a central force, i.e., it lies along the vector, $\mathbf{r}_i(t) - \mathbf{r}_j(t)$, between the two particles.

The angular momentum of the i^{th} particle is then

$$\mathbf{L}_i = (\mathbf{r}_i(t) - \mathbf{R}) \times \mathbf{p}_i(t)$$

with the torque on the i^{th} particle produced by the j^{th} given by

$$\mathbf{N}_{ij} = (\mathbf{r}_i(t) - \mathbf{R}) \times \mathbf{F}_{ij}(t)$$

The total torque on the i^{th} particle

$$\mathbf{N}_i = \sum_{j=1}^{N} (\mathbf{r}_i(t) - \mathbf{R}) \times \mathbf{F}_{ij}(t)$$

therefore satisfies,

$$\mathbf{N}_i = \frac{d\mathbf{L}_i}{dt}$$

Show that the total angular momentum of the system,

$$\mathbf{L}_{tot} = \sum_{i=1}^{N} \mathbf{L}_i$$

is conserved.