Problems

October 7, 2014

1. By finding the equation of motion, show that the Lagrangian
$$L = \frac{1}{12}m^2\dot{x}^4 + \frac{1}{2}km\dot{x}^2x^2 - \frac{1}{4}k^2x^4$$
describes the 1-dimensional simple harmonic oscillator.

2. Reproduce the scaling argument to show used for the usual harmonic oscillator Lagrangian to show that the period is given by
$$T = T_0\sqrt{\frac{m}{k}}$$
for some constant T_0.

3. Develop Noether's theorem when the Lagrangian depends on the first n time derivatives of the position $x^i(t)$,
$$S = \int_{t_1}^{t_2} L\left(\mathbf{x}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}\right)$$
where $\mathbf{x}$ (or x^i) is the position vector of a particle and $\mathbf{x}^{(k)} = \frac{d^k\mathbf{x}}{dt^k}$ for any k (or $x^i_{(k)}$).

   (a) Show that the equation of motion (extremum of S) is given by
$$\sum_{i=1}^{3}\sum_{k=0}^{n}(-1)^k\frac{d^k}{dt^k}\left(\frac{\partial L}{\partial x^i_{(k)}}\right) = 0$$

   (b) Now assume S has a symmetry $\boldsymbol{\varepsilon}(x)$, so that $S[\mathbf{x}+\boldsymbol{\varepsilon}] = S[\mathbf{x}]$ for some specific continuous variations, $\boldsymbol{\varepsilon}$. Show that the quantity
$$I = \sum_{i=1}^{3}\sum_{m=1}^{n}(-1)^{m-1}\frac{d^{m-1}}{dt^{m-1}}\frac{\partial L(x(\lambda))}{\partial x^i_{(k)}}\frac{d^{k-m}}{dt^{k-m}}\varepsilon^i(x)$$
is conserved. Answer: For this extended case a *general* variation of the action is
$$\begin{aligned}\delta S[x(t)] &\equiv \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x^i}\delta x^i + \ldots + \frac{\partial L}{\partial x^i_{(n)}}\frac{d^n}{dt^n}\delta x^i\right)dt \\ &= \int_{t_1}^{t_2}\sum_{k=0}^{n}\frac{\partial L}{\partial x^i_{(k)}}\delta x^i_{(k)}dt\end{aligned}$$
where the k^{th} term is:
$$I_k = \int_{t_1}^{t_2}\sum_{i=0}^{3}\frac{\partial L(x(\lambda))}{\partial x^i_{(k)}}\frac{d^k}{dt^k}\varepsilon^i(x)\,dt$$

Integrate this term by parts k times, keeping careful track of the surface terms. After writing the surface term for the k^{th} integral as a sum over m, sum over all k.

$$
\begin{aligned}
I_k &= \sum_{i=0}^{3} \int_{t_1}^{t_2} \frac{\partial L}{\partial x^i_{(k)}} \frac{d^k}{dt^k} \varepsilon^i dt \\
&= \sum_{i=0}^{3} \frac{\partial L}{\partial x^i_{(k)}} \frac{d^{k-1}\varepsilon^i}{dt^{k-1}} \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{i=0}^{3} \left(\frac{d}{dt} \frac{\partial L}{\partial x^i_{(k)}} \right) \frac{d^{k-1}}{dt^{k-1}} \varepsilon^i dt \\
&= \sum_{i=0}^{3} \frac{\partial L}{\partial x^i_{(k)}} \frac{d^{k-1}\varepsilon^i}{dt^{k-1}} \Bigg|_{t_1}^{t_2} - \sum_{i=0}^{3} \left(\frac{d}{dt} \frac{\partial L}{\partial x^i_{(k)}} \right) \frac{d^{k-2}\varepsilon^i}{dt^{k-2}} \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_{i=0}^{3} \left(\frac{d^2}{dt^2} \frac{\partial L}{\partial x^i_{(k)}} \right) \frac{d^{k-2}}{dt^{k-2}} \varepsilon^i dt \\
&= \sum_{m=1}^{k} \sum_{i=0}^{3} (-1)^m \left(\frac{d^{m-1}}{dt^{m-1}} \frac{\partial L}{\partial x^i_{(k)}} \right) \frac{d^{k-m}\varepsilon^i}{dt^{k-m}} \Bigg|_{t_1}^{t_2} + \sum_{i=0}^{3} (-1)^k \int_{t_1}^{t_2} \sum_{i=0}^{3} \left(\frac{d^k}{dt^k} \frac{\partial L}{\partial x^i_{(k)}} \right) \varepsilon^i dt
\end{aligned}
$$

Therefore, summing over k,

$$
\delta S\left[x\left(t\right)\right] = \sum_{k=1}^{n} \sum_{m=1}^{k} \sum_{i=0}^{3} \left((-1)^m \frac{d^{m-1}}{dt^{m-1}} \frac{\partial L}{\partial x^i_{(k)}} \right) \frac{d^{k-m}\varepsilon^i}{dt^{k-m}} \Bigg|_{t_1}^{t_2} + \sum_{k=0}^{n} \sum_{i=0}^{3} (-1)^k \int_{t_1}^{t_2} \sum_{i=0}^{3} \left(\frac{d^k}{dt^k} \frac{\partial L}{\partial x^i_{(k)}} \right) \varepsilon^i dt
$$

Imposing the field equation, the final term vanishes. If we also set ε^i to the symmetry transformation, then δS vanishes. We are left with

$$
\sum_{k=1}^{n} \sum_{m=1}^{k} \sum_{i=0}^{3} \left((-1)^m \frac{d^{m-1}}{dt^{m-1}} \frac{\partial L}{\partial x^i_{(k)}} \right) \frac{d^{k-m}\varepsilon^i}{dt^{k-m}} \Bigg|_{t_1}^{t_2} = 0
$$

Write this out for $n = 3$:

$$
\begin{aligned}
0 &= \sum_{k=1}^{3} \sum_{m=1}^{k} \sum_{i=0}^{3} \left((-1)^m \frac{d^{m-1}}{dt^{m-1}} \frac{\partial L}{\partial x^i_{(k)}} \right) \frac{d^{k-m}\varepsilon^i}{dt^{k-m}} \Bigg|_{t_1}^{t_2} \\
&= \sum_{i=0}^{3} \left((-1)^1 \frac{\partial L}{\partial x^i_{(1)}} \right) \varepsilon^i \Bigg|_{t_1}^{t_2} + \sum_{m=1}^{2} \sum_{i=0}^{3} \left((-1)^m \frac{d^{m-1}}{dt^{m-1}} \frac{\partial L}{\partial x^i_{(2)}} \right) \frac{d^{2-m}\varepsilon^i}{dt^{2-m}} \Bigg|_{t_1}^{t_2} \\
&\quad + \sum_{m=1}^{3} \sum_{i=0}^{3} \left((-1)^m \frac{d^{m-1}}{dt^{m-1}} \frac{\partial L}{\partial x^i_{(3)}} \right) \frac{d^{3-m}\varepsilon^i}{dt^{3-m}} \Bigg|_{t_1}^{t_2} \\
&= \sum_{i=0}^{3} \left(-\frac{\partial L}{\partial x^i_{(1)}} \right) \varepsilon^i \Bigg|_{t_1}^{t_2} + \sum_{i=0}^{3} \left(-\frac{\partial L}{\partial x^i_{(2)}} \right) \frac{d\varepsilon^i}{dt} \Bigg|_{t_1}^{t_2} + \sum_{i=0}^{3} \frac{d}{dt} \frac{\partial L}{\partial x^i_{(2)}} \varepsilon^i \Bigg|_{t_1}^{t_2} \\
&\quad + \sum_{i=0}^{3} \left(-\frac{\partial L}{\partial x^i_{(3)}} \right) \frac{d^2\varepsilon^i}{dt^2} \Bigg|_{t_1}^{t_2} + \sum_{i=0}^{3} \left(\frac{d}{dt} \frac{\partial L}{\partial x^i_{(3)}} \right) \frac{d\varepsilon^i}{dt} \Bigg|_{t_1}^{t_2} + \sum_{i=0}^{3} \left(-\frac{d^2}{dt^2} \frac{\partial L}{\partial x^i_{(3)}} \right) \varepsilon^i \Bigg|_{t_1}^{t_2} \\
&= -\sum_{i=0}^{3} \left(\frac{d^2}{dt^2} \frac{\partial L}{\partial x^i_{(3)}} - \frac{d}{dt} \frac{\partial L}{\partial x^i_{(2)}} + \frac{\partial L}{\partial x^i_{(1)}} \right) \varepsilon^i \Bigg|_{t_1}^{t_2} \\
&\quad + \sum_{i=0}^{3} \left(\frac{d}{dt} \frac{\partial L}{\partial x^i_{(3)}} - \frac{\partial L}{\partial x^i_{(2)}} \right) \frac{d\varepsilon^i}{dt} \Bigg|_{t_1}^{t_2} - \sum_{i=0}^{3} \frac{\partial L}{\partial x^i_{(3)}} \frac{d^2\varepsilon^i}{dt^2} \Bigg|_{t_1}^{t_2}
\end{aligned}
$$

Since the limit times are arbitrary, the expression must vanish at any time, so that

$$
I = \sum_{i=0}^{3} \left[-\left(\frac{d^2}{dt^2} \frac{\partial L}{\partial x^i_{(3)}} - \frac{d}{dt} \frac{\partial L}{\partial x^i_{(2)}} + \frac{\partial L}{\partial x^i_{(1)}} \right) \varepsilon^i + \left(\frac{d}{dt} \frac{\partial L}{\partial x^i_{(3)}} - \frac{\partial L}{\partial x^i_{(2)}} \right) \frac{d\varepsilon^i}{dt} - \frac{\partial L}{\partial x^i_{(3)}} \frac{d^2\varepsilon^i}{dt^2} \right]
$$

is a constant of the motion.The equation of motion for $n=3$ is

$$
\begin{aligned}
0 &= \sum_{i=1}^{3}\sum_{k=0}^{3}(-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial x^i_{(k)}}\right) \\
&= \sum_{i=1}^{3}\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x^i_{(1)}} + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x^i_{(2)}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial x^i_{(3)}}\right)\right)
\end{aligned}
$$

4. Suppose the Lagrangian *does* depend explicitly on time so that

$$\frac{\partial L}{\partial t} \neq 0$$

Show that energy is not conserved, but increases or decreases with time, depending on whether $\frac{\partial L}{\partial t}$ is positive or negative. Therefore, explicit time dependence in a Lagrangian gives a way to introduce dissipation or sources into a system. Since an isolated system has neither dissipation nor any outside source of energy, the energy of an isolated system is conserved.

5. Find the Euler-Lagrange equation for the following action functionals:

   (a) $S[x] = \int \exp\left(\alpha \mathbf{x}^2 + \beta \mathbf{v}^2\right) dt$ for constants α and β.

   (b) $S[x] = \int f\left(\mathbf{x}^2 \mathbf{v}^2\right) dt$ for any given function, f.

   (c) $S[x] = \frac{1}{\int \mathbf{x}\cdot\mathbf{a}dt} + \int \mathbf{x}\cdot\mathbf{a}dt$, where $\mathbf{a} = \ddot{\mathbf{x}}$.

6. Apply the techniques for generalized Euler-Lagrange systems to the following fourth-order action:

$$
\begin{aligned}
S &= \int L dt \\
&= \int \left(\frac{1}{2}km\dot{x}^2x^2 - \frac{1}{4}k^2x^4 + \frac{1}{4}m^2x\dot{x}^2\ddot{x} + \frac{1}{4}m^2x^2\ddot{x}^2 + \frac{1}{4}m^2x^2\dot{x}x^{(3)}\right) dt
\end{aligned}
$$

   (a) Find the equation of motion

   (b) Find the conserved energy. Answer: Conservation of energy follows from $\frac{\partial L}{\partial t} = 0$. For this case this holds. To find the form of the energy, write the total time derivative of L,

$$\frac{dL}{dt} = \frac{\partial L}{\partial x}x_{(1)} + \frac{\partial L}{\partial x_{(1)}}x_{(2)} + \frac{\partial L}{\partial x_{(2)}}x_{(3)} + \frac{\partial L}{\partial x_{(3)}}x_{(4)}$$

   The equation of motion from problem 3 is

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x_{(1)}} + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(2)}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial x_{(3)}}\right)$$

   so we may replace the first term, and rearrange,

$$
\begin{aligned}
\frac{dL}{dt} &= -\left(-\frac{d}{dt}\frac{\partial L}{\partial x_{(1)}} + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(2)}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial x_{(3)}}\right)\right)x_{(1)} + \frac{\partial L}{\partial x_{(1)}}x_{(2)} + \frac{\partial L}{\partial x_{(2)}}x_{(3)} + \frac{\partial L}{\partial x_{(3)}}x_{(4)} \\
&= \left(\frac{d}{dt}\frac{\partial L}{\partial x_{(1)}}x_{(1)} + \frac{\partial L}{\partial x_{(1)}}x_{(2)}\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)} + \frac{\partial L}{\partial x_{(2)}}x_{(3)} + \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)} + \frac{\partial L}{\partial x_{(3)}}x_{(4)} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(1)}}x_{(1)}\right) - \left[\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)}\right) - \left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(2)}\right)\right] \\
&\quad + \frac{\partial L}{\partial x_{(2)}}x_{(3)} + \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)} + \frac{\partial L}{\partial x_{(3)}}x_{(4)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(1)}}x_{(1)}\right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)}\right) \\
&\quad + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(2)} + \frac{\partial L}{\partial x_{(2)}}x_{(3)}\right) + \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)} + \frac{\partial L}{\partial x_{(3)}}x_{(4)} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(1)}}x_{(1)}\right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}x_{(2)}\right) \\
&\quad + \left[\frac{d}{dt}\left(\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)}\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(2)}\right] + \frac{\partial L}{\partial x_{(3)}}x_{(4)} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(1)}}x_{(1)}\right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}x_{(2)}\right) \\
&\quad + \frac{d}{dt}\left(\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)}\right) \\
&\quad - \left[\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(2)}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(3)}\right] + \frac{\partial L}{\partial x_{(3)}}x_{(4)} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(1)}}x_{(1)}\right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}x_{(2)}\right) \\
&\quad + \frac{d}{dt}\left(\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)}\right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(2)}\right) \\
&\quad + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(3)} + \frac{\partial L}{\partial x_{(3)}}x_{(4)}\right) \\
&= \frac{d}{dt}\left[\frac{\partial L}{\partial x_{(1)}}x_{(1)} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)} + \frac{\partial L}{\partial x_{(2)}}x_{(2)} + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(2)} + \frac{\partial L}{\partial x_{(3)}}x_{(3)}\right]
\end{aligned}
$$

From this we see that

$$
\begin{aligned}
E &\equiv \frac{\partial L}{\partial x_{(1)}}x_{(1)} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right)x_{(1)} + \frac{\partial L}{\partial x_{(2)}}x_{(2)} + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(1)} \\
&\quad - \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(3)}}\right)x_{(2)} + \frac{\partial L}{\partial x_{(3)}}x_{(3)} - L \\
&= \left(\frac{\partial L}{\partial x_{(1)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(2)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial x_{(3)}}\right)\right)x_{(1)} \\
&\quad + \left(\frac{\partial L}{\partial x_{(2)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_{(3)}}\right)\right)x_{(2)} + \frac{\partial L}{\partial x_{(3)}}x_{(3)} - L
\end{aligned}
$$

is conserved.

7. Consider the 3-dimensional action

$$S = \int \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - mgz\right) dt$$

where $\mathbf{x} = (x, y, z)$.

   (a) Show that there are four symmetries of S.
   (b) Find the four conserved quantities.

8. Consider the action functional for a pendulum,

$$S = \frac{1}{2}\int \left(ml^2\dot{\varphi}^2 + mgl\varphi^2\right) dt$$

Find all rescalings of the parameters and coordinates (m, g, l, φ, t) which leave S changed by no more than an overall constant. Use these rescalings to show that the period of the motion is proportional to $\sqrt{\frac{l}{g}}$.

9. The action functional
$$S = \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{\mathbf{x}}^2 - \frac{K}{\sqrt{\mathbf{x}^2}} \right) dt$$
Use a scaling argument to derive Kepler's law relating the period and a characteristic length of the orbit.

10. Using Lagrange multipliers, find the motion of a block of mass m sliding down a frictionless plane inclined at angle θ, and find the force of constraint (i.e., the force the plane exerts on the block). Then work the inclined plane problem directly from Newton's second law and check explicitly that the force applied by the plane is
$$F^i = mg \cos\theta \left(-\sin\theta, 0, \cos\theta \right)$$

11. Repeat the inclined plane problem with a moving plane, using Lagrange multipliers. Let the plane move in the direction
$$v^i = (v_1, 0, v_3)$$
Find the work done on the particle by the plane. For what velocities v^i does the particle stay in the same position on the plane?

12. A particle of mass m moves frictionlessly on the surface $z = k\rho^2$, where $\rho = \sqrt{x^2 + y^2}$ is the polar radius. Let gravity act in the $-z$ direction, $\mathbf{F} = -mg\mathbf{k}$. Use a Lagrange multiplier to impose the constraint and find the motion of the system.

13. A ball moves frictionlessly on a horizontal tabletop. The ball of mass m is connected to a string of length L which passes through a hole in the tabletop and is fastened to a pendulum of mass M. The string is free to slide through the hole in either direction. Use Lagrange multipliers to impose the constraints. Find the motion of the ball and pendulum.

14. Study the motion of a spherical pendulum: a negligably light rod of length L with a mass m attached to one end. The remaining end is fixed in space. The mass is therefore free to move anywhere on the surface of a sphere of radius L, under the influence of gravity, $-mg\mathbf{k}$.

    (a) Write the Lagrangian for the system, imposing any constraints with Lagrange multipliers.

    (b) Identify any conserved quantities.

    (c) Use the conservation laws and any needed equations of motion to solve for the motion. In particular study the following motions:

        i. Motion confined to a vertical plane, of small amplitude.

        ii. Motion confined to a vertical plane, of arbitrary amplitude.

        iii. Motion confined to a horizontal plane.

    (d) Beginning with your solution for motion in a horizontal plane, study small oscillations away from the plane.