Hamiltonian Mechanics

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1 Phase space

Phase space is a dynamical arena for classical mechanics in which the number of independent dynamical variables is doubled from n variables q_i, $i = 1, 2, \ldots, n$ to $2n$ by treating either the velocities or the momenta as independent variables. This has two important consequences.

First, the equations of motion become first order differential equations instead of second order, so that the initial conditions is enough to specify a unique point in phase space. The means that, unlike the configurations space treatment, there is a unique solution to the equations of motion through each point. This permits some useful geometric techniques in the study of the system.

Second, as we shall see, the set of transformations that preserve the equations of motion is enlarged. In Lagrangian mechanics, we are free to use n general coordinates, q_i, for our description. In phase space, however, we have $2n$ coordinates. Even though transformations among these $2n$ coordinates are not completely arbitrary, there are far more allowed transformations. This large set of transformations allows us, in principal, to formulate a general solution to mechanical problems via the Hamilton-Jacobi equation.

1.1 Velocity phase space

While we will not be using velocity phase space here, it provides some motivation for our developments in the next Sections. The formal presentation of Hamiltonian dynamics begins in Section 1.3.

Suppose we have an action functional

$$S = \int L\left(q_i, \dot{q}_j, t\right) dt$$

dependent on n dynamical variables, $q_i(t)$, and their time derivatives. We might instead treat $L(q_i, u_j, t)$ as a function of $2n$ dynamical variables. Thus, instead of treating the the velocities as time derivatives of the position variables, $(q_i, \dot{q}_i)$ we introduce n velocities u_i and treat them as independent. Then the variations of the velocities δu_i are also independent, and we end up with $2n$ equations. Finally, we include n constraints, restoring the relationship between q_i and $\dot{q}_i$,

$$S = \int \left[L\left(q_i, u_j, t\right) + \sum \lambda_i\left(\dot{q}_i - u_i\right)\right] dt$$

Variation of the original dynamical variables results in

$$\begin{aligned} 0 &= \delta_q S \\ &= \int \left(\frac{\partial L}{\partial q_i}\delta q_i + \lambda_i \delta \dot{q}_i\right) dt \\ &= \int \left(\frac{\partial L}{\partial q_i} - \dot{\lambda}_i\right)\delta q_i dt \end{aligned}$$

so that

$$\dot{\lambda}_i = \frac{\partial L}{\partial q_i}$$

For the velocities, we find

$$\begin{aligned} 0 &= \delta_u S \\ &= \int \left(\frac{\partial L}{\partial u_i} - \lambda_i \right) \delta u_i dt \end{aligned}$$

so that

$$\lambda_i = \frac{\partial L}{\partial u_i}$$

and finally, varying the Lagrange multipliers, λ_i, we recover the constraints,

$$u_i = \dot{q}_i$$

We may eliminate the multipliers by differentiating the velocity equation

$$\frac{d}{dt}(\lambda_i) = \frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right)$$

to find $\dot{\lambda}_i$, then substituting for u_i and $\dot{\lambda}_i$ into the q_i equation,

$$\begin{aligned} \dot{\lambda}_i &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) &= \frac{\partial L}{\partial q_i} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0 \end{aligned}$$

and we recover the Euler-Lagrange equations. If the kinetic energy is of the form $\sum_{i=1}^{n} \frac{1}{2} m u_i^2$, then the Lagrange multipliers are just the momenta,

$$\begin{aligned} \lambda_i &= \frac{\partial L}{\partial u_i} \\ &= m u_i \\ &= m \dot{q}_i \end{aligned}$$

1.2 Phase space

We can make the construction above more general by requiring the Lagrange multipliers to always be the conjugate momentum. Combining the constraint equation with the equation for λ_i we have

$$\lambda_i = \frac{\partial L}{\partial \dot{q}_i}$$

We now define the conjugate momentum to be exactly this derivative,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

Then the action becomes

$$\begin{aligned} S &= \int \left[L(q_i, u_j, t) + \sum p_i \dot{q}_i - \sum p_i u_i \right] dt \\ &= \int \left[L(q_i, u_j, t) - \sum p_i u_i + \sum p_i \dot{q}_i \right] dt \end{aligned}$$

For Lagrangians quadratic in the velocities, the first two terms become

$$\begin{aligned} L(q_i, u_j, t) - \sum p_i u_i &= L(q_i, \dot{q}, t) - \sum p_i \dot{q}_i \\ &= T - V - \sum p_i \dot{q}_i \\ &= -(T + V) \end{aligned}$$

We define this quantity to be the Hamiltonian,

$$H \equiv \sum p_i \dot{q}_i - L(q_i, \dot{q}, t)$$

Then

$$S = \int \left[\sum p_i \dot{q}_i - H \right] dt$$

This successfully eliminates the Lagrange multipliers from the formulation.

The term "phase space" is generally reserved for momentum phase space, spanned by coordinates q_i, p_j.

1.2.1 Legendre transformation

Notice that $H = \sum p_j \dot{q}_j - L$ is, by definition, independent of the velocities, since

$$\begin{aligned} \frac{\partial H}{\partial \dot{q}_i} &= \frac{\partial}{\partial \dot{q}_i} \left(\sum_j p_j \dot{q}_j - L \right) \\ &= \sum_j p_j \delta_{ij} - \frac{\partial L}{\partial \dot{q}_i} \\ &= p_i - \frac{\partial L}{\partial \dot{q}_i} \\ &\equiv 0 \end{aligned}$$

Therefore, the Hamiltonian is a function of q_i and p_i only. This is an example of a general technique called Legendre transformation. Suppose we have a function f, which depends on independent variables A, B and dependent variables, having partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial A} &= P \\ \frac{\partial f}{\partial B} &= Q \end{aligned}$$

Then the differential of f is

$$df = PdA + QdB$$

A Legendre transformation allows us to interchange variables to make either P or Q or both into the independent variables. For example, let $g(A, B, P) \equiv f - PA$. Then

$$\begin{aligned} dg &= df - AdP - PdA \\ &= PdA + QdB - AdP - PdA \\ &= QdB - AdP \end{aligned}$$

so that g actually only changes with B and P, $g = g(B, P)$. Similarly, $h = f - QB$ is a function of (A, Q) only, while $k = -(f - PA - QB)$ has (P, Q) as independent variables. Explicitly,

$$\begin{aligned} dk &= -df + PdA + AdP + QdB + BdQ \\ &= AdP + BdQ \end{aligned}$$

and we now have

$$\begin{aligned}\frac{\partial f}{\partial P} &= A \\ \frac{\partial f}{\partial Q} &= B\end{aligned}$$

Legendre transformations are familiar from thermodynamics, where the internal energy $U(S,V)$ is given by the second law,

$$dU = TdS - PdV$$

It may be altered to give the Helmholz free energy, $A = U - TS$, the enthalpy, $H(S,P) = U + PV$, or the the Gibbs free energy, $g(T,P) = U - TS + PV$.

2 Hamilton's equations

The essential formalism of Hamiltonian mechanics is as follows. We begin with the action

$$S = \int L(q_i, \dot{q}_j, t)\, dt$$

and define the conjugate momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

and Hamiltonian

$$H(q_i, p_j, t) \equiv \sum p_j \dot{q}_j - L(q_i, \dot{q}_j, t)$$

Then the action may be written as

$$S = \int \left[\sum p_j \dot{q}_j - H(q_i, p_j, t)\right] dt$$

where q_i and p_j are now treated as independent variables.

Finding extrema of the action with respect to all $2n$ variables, we find:

$$\begin{aligned}0 &= \delta_{q_k} S \\ &= \int \left(\left(\frac{\partial}{\partial \dot{q}_k}\sum_j p_j \dot{q}_j\right)\delta\dot{q}_k - \frac{\partial H}{\partial q_k}\delta q_k\right) dt \\ &= \int \left(\sum_j p_j \delta_{jk}\delta\dot{q}_k - \frac{\partial H}{\partial q_k}\delta q_k\right) dt \\ &= \int \left(p_k \delta\dot{q}_k - \frac{\partial H}{\partial q_k}\delta q_k\right) dt \\ &= \int \left(-\dot{p}_k - \frac{\partial H}{\partial q_k}\right)\delta q_k dt\end{aligned}$$

so that

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}$$

and

$$
\begin{aligned}
0 &= \delta_{pk} S \\
&= \int \left(\left(\frac{\partial}{\partial p_k} \sum_j p_j \dot{q}_j \right) \delta p_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt \\
&= \int \left(\dot{q}_k \delta p_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt \\
&= \int \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k dt
\end{aligned}
$$

so that

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

These are Hamilton's equations. Whenever the Legendre transformation between L and H and between $\dot{q}_k$ and p_k is non-degenerate, Hamilton's equations,

$$
\begin{aligned}
\dot{q}_k &= \frac{\partial H}{\partial p_k} \\
\dot{p}_k &= -\frac{\partial H}{\partial q_k}
\end{aligned}
$$

form a system equivalent to the Euler-Lagrange or Newtonian equations.

2.1 Example: Newton's second law

Suppose the Lagrangian takes the form

$$L = \frac{1}{2} m \dot{\mathbf{x}}^2 - V(\mathbf{x})$$

Then the conjugate momenta are

$$
\begin{aligned}
p_i &= \frac{\partial L}{\partial \dot{x}_i} \\
&= m \dot{x}_i
\end{aligned}
$$

and the Hamiltonian becomes

$$
\begin{aligned}
H(x_i, p_j, t) &\equiv \sum p_j \dot{x}_j - L(x_i, \dot{x}_j, t) \\
&= m\dot{\mathbf{x}}^2 - \frac{1}{2} m \dot{\mathbf{x}}^2 + V(\mathbf{x}) \\
&= \frac{1}{2} m \dot{\mathbf{x}}^2 + V(\mathbf{x}) \\
&= \frac{1}{2m} \mathbf{p}^2 + V(\mathbf{x})
\end{aligned}
$$

Notice that we must invert the relationship between the momenta and the velocities,

$$\dot{x}_i = \frac{p_i}{m}$$

then expicitly replace all occurrences of the velocity with appropriate combinations of the momentum.

Hamilton's equations are:

$$\begin{aligned}
\dot{x}_k &= \frac{\partial H}{\partial p_k} \\
&= \frac{p_k}{m} \\
\dot{p}_k &= -\frac{\partial H}{\partial x_k} \\
&= -\frac{\partial V}{\partial x_k}
\end{aligned}$$

thereby reproducing the usual definition of momentum and Newton's second law.

2.2 Example: coupled oscillators

Suppose we have coupled oscillators comprised of two identical pendula of length l and each of mass m, connected by a light spring with spring constant k. Then since the potential of the spring is

$$\begin{aligned}
\frac{1}{2}k\left(\triangle x^2 + \triangle y^2\right) &= \frac{1}{2}k\left((l\sin\theta_1 - l\sin\theta_2)^2 + (l\cos\theta_1 - l\cos\theta_2)^2\right) \\
&= \frac{1}{2}kl^2\left(\sin^2\theta_1 - 2\sin\theta_1\sin\theta_2 + \sin^2\theta_2 + \cos^2\theta_1 - 2\cos\theta_1\cos\theta_2 + \cos^2\theta_2^2\right) \\
&= kl^2\left(1 - \sin\theta_1\sin\theta_2 - \cos\theta_1\cos\theta_2\right) \\
&= kl^2\left(1 - \cos\left(\theta_1 - \theta_2\right)\right)
\end{aligned}$$

the action becomes

$$S = \int \left[\frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - kl^2\left(1 - \cos\left(\theta_1 - \theta_2\right)\right) - mgl\left(1 - \cos\theta_1\right) - mgl\left(1 - \cos\theta_2\right)\right] dt$$

which for small angles becomes approximately

$$S = \int \left[\frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2}kl^2\left(\theta_1 - \theta_2\right)^2 - \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right)\right] dt$$

The conjugate momenta are:

$$\begin{aligned}
p_1 &= \frac{\partial L}{\partial \dot{\theta}_1} \\
&= ml^2\dot{\theta}_1 \\
p_2 &= \frac{\partial L}{\partial \dot{\theta}_2} \\
&= ml^2\dot{\theta}_2
\end{aligned}$$

and the Hamiltonian is

$$\begin{aligned}
H &= p_1\dot{\theta}_1 + p_2\dot{\theta}_2 - L \\
&= ml^2\dot{\theta}_1^2 + ml^2\dot{\theta}_2^2 - \left(\frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2}kl^2\left(\theta_1 - \theta_2\right)^2 - \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right)\right) \\
&= \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) + \frac{1}{2}kl^2\left(\theta_1 - \theta_2\right)^2 + \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right) \\
&= \frac{1}{2ml^2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}kl^2\left(\theta_1 - \theta_2\right)^2 + \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right)
\end{aligned}$$

Notice again our elimination of the velocities in favor of the momenta.

Hamilton's equations are:

$$\begin{aligned}
\dot{\theta}_1 &= \frac{\partial H}{\partial p_1} \\
&= \frac{1}{ml^2} p_1 \\
\dot{\theta}_2 &= \frac{\partial H}{\partial p_2} \\
&= \frac{1}{ml^2} p_2 \\
\dot{p}_1 &= -\frac{\partial H}{\partial \theta_1} \\
&= -\frac{1}{2} kl^2 (\theta_1 - \theta_2) - mgl\theta_1 \\
\dot{p}_2 &= -\frac{\partial H}{\partial \theta_2} \\
&= \frac{1}{2} kl^2 (\theta_1 - \theta_2) - mgl\theta_2
\end{aligned}$$

From here we may solve in any way that suggests itself. If we differentiate $\dot{\theta}_1$ again, and use the third equation, we have

$$\begin{aligned}
\ddot{\theta}_1 &= \frac{1}{ml^2} \dot{p}_1 \\
&= -\frac{k}{2m} (\theta_1 - \theta_2) - \frac{g}{l} \theta_1
\end{aligned}$$

Similarly, for θ_2 we have

$$\ddot{\theta}_2 = \frac{k}{2m} (\theta_1 - \theta_2) - \frac{g}{l} \theta_2$$

Subtracting,

$$\begin{aligned}
\ddot{\theta}_1 - \ddot{\theta}_2 &= -\frac{k}{m} (\theta_1 - \theta_2) - \frac{g}{l} (\theta_1 - \theta_2) \\
\frac{d^2}{dt^2} (\theta_1 - \theta_2) + \left(\frac{k}{m} + \frac{g}{l} \right) (\theta_1 - \theta_2) &= 0
\end{aligned}$$

so that

$$\theta_1 - \theta_2 = A \sin \omega_1 t + B \cos \omega_1 t$$

with

$$\omega_1 = \sqrt{\frac{k}{m} + \frac{g}{l}}$$

Adding instead, we find

$$\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{l} (\theta_1 + \theta_2)$$

so that

$$\theta_1 + \theta_2 = C \sin \omega_2 t + D \cos \omega_2 t$$

where $\omega_2 = \sqrt{\frac{g}{l}}$. Notice that ω_2 depends only on the gravitational restoring force since changing the total angle $\theta_1 + \theta_2$ does not stretch the spring.

The general motion is therefore a sum of two simple harmonic motions, with frequencies ω_1 and ω_2.

$$\begin{aligned}
\theta_1 &= \frac{1}{2}\left(A\sin\omega_1 t + B\cos\omega_1 t + C\sin\omega_2 t + D\cos\omega_2 t\right) \\
\theta_2 &= \frac{1}{2}\left(-A\sin\omega_1 t - B\cos\omega_1 t + C\sin\omega_2 t + D\cos\omega_2 t\right) \\
p_1 &= \frac{1}{2}ml^2\left(A\omega_1\cos\omega_1 t - B\omega_2\sin\omega_1 t + C\omega_2\cos\omega_2 t - D\omega_2\sin\omega_2 t\right) \\
p_2 &= \frac{1}{2}ml^2\left(-A\omega_1\cos\omega_1 t + B\omega_2\sin\omega_1 t + C\omega_2\cos\omega_2 t - D\omega_2\sin\omega_2 t\right)
\end{aligned}$$

The constants A, B, C, D are determined by the four initial conditions θ_{i0} and p_{i0} at time $t = 0$ by solving:

$$\begin{aligned}
\theta_{10} &= \frac{1}{2}(B + D) \\
\theta_{20} &= \frac{1}{2}(D - B) \\
p_{10} &= ml^2(\omega_1 A + \omega_2 C) \\
p_{20} &= ml^2(\omega_2 C - \omega_1 A)
\end{aligned}$$

This results in

$$\begin{aligned}
A &= \frac{p_{10}}{2ml^2\omega_1} - \frac{p_{20}}{2ml^2\omega_1} \\
B &= \theta_{10} - \theta_{20} \\
C &= \frac{p_{10}}{2ml^2\omega_2} + \frac{p_{20}}{2ml^2\omega_2} \\
D &= \theta_{10} + \theta_{20}
\end{aligned}$$

Notice that only one phase space curve, $(\theta_1, \theta_2, p_1, p_2)$ passes through the phase space point $(\theta_{10}, \theta_{20}, p_{10}, p_{20})$.