Perihelion advance in modified Newtonian potentials

November 12, 2014

Let the Newtonian gravitational potential be modified to

$$V(r) = -\frac{\alpha}{r} f(r)$$

Suppose we can expand f in inverse powers of

$$f(r) = \sum_{n=0}^{\infty} \frac{a_n}{r^n}$$

with $a_0 = 1$. We keep the first two new terms,

$$f \approx 1 + \frac{a}{r} + \frac{b}{r^2}$$

We still have the energy equation and conserved angular momentum,

$$\begin{aligned} E &= \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{\alpha}{r} f(r) \\ L &= \mu r^2 \dot{\varphi} \end{aligned}$$

Circular orbits occur at minima of the effective potential,

$$\begin{aligned} V_{eff} &= \frac{L^2}{2\mu r_0^2} - \frac{\alpha}{r_0}\left(1 + \frac{a}{r_0} + \frac{b}{r_0^2}\right) \\ 0 &= \frac{dV_{eff}}{dr} \\ &= -\frac{L^2}{\mu r_0^3} + \frac{\alpha}{r_0^2}\left(1 + \frac{a}{r_0} + \frac{b}{r_0^2}\right) - \frac{\alpha}{r_0}\left(-\frac{a}{r_0^2} - \frac{2b}{r_0^3}\right) \\ &= -\frac{L^2}{\mu r_0^3} + \frac{\alpha}{r_0^2} + \frac{2\alpha a}{r_0^3} + \frac{3\alpha b}{r_0^4} \\ \frac{L^2}{\mu} &= \alpha r_0\left(1 + \frac{a}{r_0} + \frac{b}{r_0^2}\right) - \alpha r_0^2\left(-\frac{a}{r_0^2} - \frac{2b}{r_0^3}\right) \\ &= \alpha r_0 + 2\alpha a + \frac{3\alpha b}{r_0} \end{aligned}$$

The energy and angular momentum for the circular orbit are

$$\begin{aligned} E &= \frac{L^2}{2\mu r_0^2} - \frac{\alpha}{r_0}\left(1 + \frac{a}{r_0} + \frac{b}{r_0^2}\right) \\ &= \frac{1}{2r_0^2}\left(\alpha r_0 + 2\alpha a + \frac{3\alpha b}{r_0}\right) - \frac{\alpha}{r_0}\left(1 + \frac{a}{r_0} + \frac{b}{r_0^2}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{r_0}\left(\frac{1}{2}+\frac{a}{r_0}+\frac{3b}{2r_0^2}-1-\frac{a}{r_0}-\frac{b}{r_0^2}\right) \\
&= -\frac{\alpha}{2r_0}\left(1-\frac{b}{r_0^2}\right) \\
\frac{L^2}{\mu} &= \alpha r_0 + 2\alpha a + \frac{3\alpha b}{r_0}
\end{aligned}
$$

so the radius determines both constants of the motion. The orbital frequency is

$$
\begin{aligned}
\dot{\varphi} &= \frac{L}{\mu r_0^2} \\
&= \frac{1}{\sqrt{\mu} r_0^2}\sqrt{\alpha r_0 + 2\alpha a + \frac{3\alpha b}{r_0}}
\end{aligned}
$$

The second derivative of the effective potential is

$$
\begin{aligned}
\frac{d^2 V_{eff}}{dr^2} &= \frac{3L^2}{\mu r_0^4} - \frac{2\alpha}{r_0^3} - \frac{6\alpha a}{r_0^4} - \frac{12\alpha b}{r_0^5} \\
&= \frac{3\alpha}{r_0^3} + \frac{6\alpha a}{r_0^4} + \frac{9\alpha b}{r_0^5} - \frac{2\alpha}{r_0^3} - \frac{6\alpha a}{r_0^4} - \frac{12\alpha b}{r_0^5} \\
&= \frac{\alpha}{r_0^3}\left(1 - \frac{3b}{r_0^2}\right)
\end{aligned}
$$

so as long as b is small, we have a minimum.

Next consider orbits which are nearly circular. Now r will oscillate between turning points. We still have

$$
\begin{aligned}
E &= \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{\alpha}{r} f(r) \\
L &= \mu r^2 \dot{\varphi}
\end{aligned}
$$

The minimum is still given by

$$
\frac{L^2}{\mu} = \alpha r_0 + 2\alpha a + \frac{3\alpha b}{r_0}
$$

and we expand r about r_0 as $r = r_0 + \varepsilon$ with $\frac{\varepsilon}{r_0} \ll 1$. Then the energy is

$$
\begin{aligned}
E &= \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{L^2}{2\mu(r_0+\varepsilon)^2} - \frac{\alpha}{r_0+\varepsilon}\left(1 + \frac{a}{r_0+\varepsilon} + \frac{b}{(r_0+\varepsilon)^2}\right) \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{\alpha r_0 + 2\alpha a + \frac{3\alpha b}{r_0}}{2r_0^2\left(1+\frac{\varepsilon}{r_0}\right)^2} - \frac{\alpha}{r_0\left(1+\frac{\varepsilon}{r_0}\right)}\left(1 + \frac{a}{r_0\left(1+\frac{\varepsilon}{r_0}\right)} + \frac{b}{r_0^2\left(1+\frac{\varepsilon}{r_0}\right)^2}\right) \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 - \frac{\alpha}{r_0\left(1+\frac{\varepsilon}{r_0}\right)} + \frac{\alpha r_0 + 2\alpha a - 2\alpha a + \frac{3\alpha b}{r_0}}{2r_0^2\left(1+\frac{\varepsilon}{r_0}\right)^2} - \frac{\alpha b}{r_0^3\left(1+\frac{\varepsilon}{r_0}\right)^3} \\
&= \frac{1}{2}\mu\dot{\varepsilon}^2 - \frac{\alpha}{r_0\left(1+\frac{\varepsilon}{r_0}\right)} + \frac{\alpha r_0 + \frac{3\alpha b}{r_0}}{2r_0^2\left(1+\frac{\varepsilon}{r_0}\right)^2} - \frac{\alpha b}{r_0^3\left(1+\frac{\varepsilon}{r_0}\right)^3}
\end{aligned}
$$

We need

$$
(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2
$$

so up to second order in $\frac{\varepsilon}{r_0}$ we have

$$\begin{aligned} E &= \frac{1}{2}\mu\dot{\varepsilon}^2 - \frac{\alpha}{r_0}\left(1 - \frac{\varepsilon}{r_0} + \left(\frac{\varepsilon}{r_0}\right)^2\right) + \left(\frac{\alpha}{2r_0} + \frac{3\alpha b}{2r_0^3}\right)\left(1 - \frac{2\varepsilon}{r_0} + 3\left(\frac{\varepsilon}{r_0}\right)^2\right) - \frac{\alpha b}{r_0^3}\left(1 - \frac{3\varepsilon}{r_0} + 6\left(\frac{\varepsilon}{r_0}\right)^2\right) \\ &= \frac{1}{2}\mu\dot{\varepsilon}^2 - \frac{\alpha}{2r_0}\left(1 - \frac{b}{r_0^2}\right) + \left(\frac{\alpha}{r_0} + \frac{3\alpha b}{r_0^3} - \frac{\alpha}{r_0} - \frac{3\alpha b}{r_0^3}\right)\frac{\varepsilon}{r_0} + \left(\frac{3\alpha}{2r_0} + \frac{9\alpha b}{2r_0^3} - \frac{6\alpha b}{r_0^3} - \frac{\alpha}{r_0}\right)\left(\frac{\varepsilon}{r_0}\right)^2 \\ &= \frac{1}{2}\mu\dot{\varepsilon}^2 + E_0 + \frac{\alpha}{2r_0}\left(1 - \frac{3b}{r_0^2}\right)\left(\frac{\varepsilon}{r_0}\right)^2 \end{aligned}$$

Therefore, the energy is modified, with the increase given by

$$E - E_0 = \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{1}{2}\frac{\alpha}{r_0^3}\left(1 - \frac{3b}{r_0^2}\right)\varepsilon^2$$

Setting

$$k = \frac{\alpha}{r_0^3}\left(1 - \frac{3b}{r_0^2}\right)$$

this is just the energy of a simple harmonic oscillator

$$E - E_0 = \frac{1}{2}\mu\dot{\varepsilon}^2 + \frac{1}{2}k\varepsilon^2$$

with frequency

$$\begin{aligned} \omega &= \sqrt{\frac{k}{\mu}} \\ &= \sqrt{\frac{\alpha}{\mu r_0^3}\left(1 - \frac{3b}{r_0^2}\right)} \end{aligned}$$

Comparing frequencies,

$$\begin{aligned} \frac{\omega}{\dot{\varphi}} &= \frac{\sqrt{\mu r_0^4}\sqrt{\frac{\alpha}{\mu r_0^3}\left(1 - \frac{3b}{r_0^2}\right)}}{\sqrt{\alpha r_0 + 2\alpha a + \frac{3\alpha b}{r_0}}} \\ &= \sqrt{\frac{\alpha r_0\left(1 - \frac{3b}{r_0^2}\right)}{\alpha r_0\left(1 + \frac{2a}{r_0} + \frac{3b}{r_0^2}\right)}} \\ &= \left(1 - \frac{3b}{r_0^2}\right)^{1/2}\left(1 + \frac{2a}{r_0} + \frac{3b}{r_0^2}\right)^{-1/2} \\ &= \left(1 - \frac{3b}{2r_0^2}\right)\left(1 - \frac{1}{2}\left(\frac{2a}{r_0} + \frac{3b}{r_0^2}\right) + \frac{3}{8}\left(\frac{2a}{r_0}\right)^2\right) \\ &= \left(1 - \frac{3b}{2r_0^2} - \frac{1}{2}\left(\frac{2a}{r_0} + \frac{3b}{r_0^2}\right) + \frac{3}{8}\left(\frac{2a}{r_0}\right)^2\right) \\ &= 1 - \frac{a}{r_0} + \frac{3}{2r_0^2}\left(a^2 - 2b\right) \end{aligned}$$

and this cannot be rational for general r_0. For $a > 0$, we have the orbital frequency $\dot{\varphi}$ faster than the frequency of radial oscillations ω, so the perihelion comes later with each orbit.