Noether's Theorem

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We continue looking at consequences of Noether's theorem, applying it to 2-dimensional rotations, scalings, and time translations. Recal that the conserved quantity for an infinitesimal symmetry $\delta x^i = \varepsilon^i(x)$ is

$$I = \frac{\partial L(x(\lambda))}{\partial \dot{x}^i} \varepsilon^i(x)$$

1 Rotational symmetry and conservation of angular momentum (2 dim)

Consider a 2-dimensional system with free-particle Lagrangian

$$L(x,y) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right)$$

The rotation

$$\begin{aligned} x &\rightarrow x' = x\cos\theta - y\sin\theta \\ y &\rightarrow y' = x\sin\theta + y\cos\theta \end{aligned}$$

for any fixed value of θ leaves the action unchanged,

$$S[\mathbf{x}] = \int L dt$$

invariant.

For an infinitesimal change, $\theta \ll 1$, the changes in x, y are

$$\begin{aligned} \varepsilon^1 &= \delta x \\ &= x' - x \\ &= x\cos\theta - y\sin\theta - x \\ &= x\left(1 - \frac{1}{2!}\theta^2 + \ldots\right) - y\left(\theta - \frac{1}{3!}\theta^3 + \ldots\right) - x \\ &\approx -y\theta \\ \varepsilon^2 &= \delta y \\ &= y' - y \\ &\approx x\theta \end{aligned}$$

Therefore, from Noether's theorem, we have the conserved quantity,

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}^i}\varepsilon^i(x) &= m\dot{x}\varepsilon^1 + m\dot{y}\varepsilon^2 \\ &= m\dot{x}(-y\theta) + m\dot{y}(x\theta) \\ &= \theta m(x\dot{y} - y\dot{x}) \end{aligned}$$

as long as x and y satisfy the equations of motion. Since θ is just an arbitrary constant to begin with, we can identify the angular momentum,

$$J \equiv m(\dot{y}x - \dot{x}y)$$

as the conserved quantity.

It is worth noting that J is conjugate to a cyclic coordinate. If we rewrite the action in terms of polar coordinates, (r, φ), it becomes

$$S[r, \varphi] = \int_{t_1}^{t_2} \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$$

so that φ is cyclic. The momentum conjugate to φ is

$$\begin{aligned} p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} \\ &= mr^2\dot{\varphi} \end{aligned}$$

Since $\tan\varphi = \frac{y}{x}$,

$$\begin{aligned} \frac{1}{\cos^2\varphi}\dot{\varphi} &= \frac{\dot{y}}{x} - \frac{y\dot{x}}{x^2} \\ \dot{\varphi} &= \frac{x\dot{y} - y\dot{x}}{x^2}\cos^2\varphi \\ &= \frac{x\dot{y} - y\dot{x}}{x^2}\left(\frac{x^2}{r^2}\right) \end{aligned}$$

giving the same result,

$$p_\varphi = mr^2\dot{\varphi} = m(x\dot{y} - y\dot{x}) = J$$

We will generalize this result to 3-dimensions after a complete discussion of rotations.

2 Conservation of energy

Conservation of energy is related to *time translation invariance*. However, this invariance is more subtle than simply replacing $t \to t + \tau$, which is simply a reparameterization of the action integral. Instead, the conservation law holds whenever the Lagrangian does not depend *explicitly* on time so that

$$\frac{\partial L}{\partial t} = 0$$

The total time derivative of L then reduces to

$$\begin{aligned} \frac{dL}{dt} &= \sum_i \frac{\partial L}{\partial x^i}\dot{x}^i + \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i + \frac{\partial L}{\partial t} \\ &= \sum_i \frac{\partial L}{\partial x^i}\dot{x}^i + \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i \end{aligned}$$

Using the Lagrange equations to replace

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}$$

in the first term, we get

$$\begin{aligned} \frac{dL}{dt} &= \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i + \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i\right) \\ &= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i\right) \end{aligned}$$

Bringing both terms to the same side, we have

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L\right) = 0$$

so that the quantity

$$E \equiv \sum_i \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L$$

is conserved. The quantity E is called the *energy*.

For a single particle in a potential $V(\mathbf{x})$, the conserved energy is

$$\begin{aligned}
E &\equiv \sum_i \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L \\
&= \sum_i \dot{x}^i \frac{\partial}{\partial \dot{x}^i}\left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) \\
&= \sum_i m\dot{x}^i\dot{x}^i - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) \\
&= \frac{1}{2}m\dot{\mathbf{x}}^2 + V(\mathbf{x})
\end{aligned}$$

For the velocity-dependent potential of the Lorentz force law,

$$S[\mathbf{x}] = \int_{t_1}^{t_2}\left[\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right]dt$$

so that

$$\begin{aligned}
E &= \sum_i \dot{x}^i \frac{\partial}{\partial \dot{x}^i}\left(\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right) \\
&= \sum_i \dot{x}^i\left(m\dot{x}^i + qA^i\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right) \\
&= \frac{1}{2}m\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}}\cdot\mathbf{A} - (-q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}) \\
&= \frac{1}{2}m\dot{\mathbf{x}}^2 + q\phi
\end{aligned}$$

is conserved.

3 Scale Invariance

As we have noted, physical measurements are always relative to our choice of unit. The resulting dilatational symmetry will be examined in detail when we study Hamiltonian dynamics. However, there are other forms of rescaling a problem that lead to physical results. These results typically depend on the fact that the Euler-Lagrange equation is unchanged by an overall constant, so that the actions

$$\begin{aligned}
S &= \int L dt \\
S' &= \alpha \int L dt
\end{aligned}$$

have the same extremal curves.

Now suppose we have a Lagrangian which depends on some constant parameters $(a_1, \ldots, a_n)$ in addition to the arbitrary coordinates,

$$L = L\left(x^i, \dot{x}^i, a_1, \ldots, a_n, t\right)$$

These parameters might include masses, lengths, spring constants and so on. Further, suppose that each of these variables may be rescaled by some factor in such a way that S changes by only an overall factor. That is, when we make the replacements

$$\begin{aligned} x^i &\rightarrow \alpha x^i \\ t &\rightarrow \beta t \\ \dot{x}^i &\rightarrow \frac{\alpha}{\beta}\dot{x}^i \\ a_i &\rightarrow \gamma_i a_i \end{aligned}$$

for certain constants $(\alpha, \beta, \gamma_1, \ldots, \gamma_n)$ we find that

$$L\left(\alpha x^i, \frac{\alpha}{\beta}\dot{x}^i, \gamma_1 a_1, \ldots, \gamma_n a_n, \beta t\right) = \sigma L\left(x^i, \dot{x}^i, a_1, \ldots, a_n, t\right)$$

for some constant σ which depends on the scaling constants. Then the Euler-Lagrange equations for the system described by $L\left(\alpha x^i, \frac{\alpha}{\beta}\dot{x}^i, \gamma_1 a_1, \ldots, \gamma_n a_n, \beta t\right)$ are the same as for the original Lagrangian, and we may make the replacements in the solution.

Consider the simple harmonic oscillator. The usual Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

If we rescale,

$$\begin{aligned} \tilde{x} &= \alpha x \\ \tilde{m} &= \beta m \\ \tilde{k} &= \gamma k \\ \tilde{t} &= \delta t \end{aligned}$$

then

$$\tilde{L} = \frac{1}{2}\frac{\beta\alpha^2}{\delta^2}m\dot{x}^2 - \frac{1}{2}\gamma\alpha^2 kx^2$$

so as long as $\frac{\beta}{\delta^2} = \gamma$ we have $\tilde{S} = \gamma\alpha^2 S$ as a scaling symmetry. Scaling x doesn't depend on the other scales, so there's no information there.

Now consider a system with unit mass and unit spring constant,

$$\begin{aligned} m_0 &= 1 \\ k_0 &= 1 \end{aligned}$$

and suppose this system is periodic, with period T_0. Then rescaling, the mass, spring constant and period become

$$\begin{aligned} m &= \beta m_0 = \beta \\ k &= \gamma k_0 = \gamma \\ T &= \delta T_0 \end{aligned}$$

and scale invariance tells us that a periodic solution also holds for the scaled m, k and T as long as $\delta = \sqrt{\frac{\beta}{\gamma}}$. Therefore

$$T = T_0\sqrt{\frac{m}{k}}$$

and the frequency is proportional to $\sqrt{\frac{k}{m}}$.