# Noether's Theorem

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There are important general properties of Euler-Lagrange systems based on the symmetry of the Lagrangian. The most important symmetry result is Noether's Theorem, which we prove be;pw. We then apply the theorem in several important special cases to find conservation of momentum, energy and angular momentum.

## 1 Noether's theorem for the Euler-Lagrange equation

In essence, Noether's theorem states that when an action has a symmetry, we can derive a conserved quantity. To prove the theorem, we need clear definitions of a symmetry and a conserved quantity.

Def: Conserved quantities We have shown that the action

$$S\left[\mathbf{x}\left(t\right)\right] = \int_{C} L\left(x^{i}, \dot{x}^{i}, t\right) dt$$

is extremal when  $x^{i}(t)$  satisfies the Euler-Lagrange equation,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = 0 \tag{1}$$

This condition guarantees that  $\delta S$  vanishes for all variations,  $x^i(t) \to x^i(t) + \delta x^i(t)$  which vanish at the endpoints of the motion. Let  $x^i(t)$  be a solution to the Euler-Lagrange equation, eq.(1) of motion. Then a function of  $x^i(t)$  and its time derivatives,

$$f\left(x^{i}\left(t\right),\dot{x}^{i}\left(t\right)\ldots,
ight)$$

is conserved if it is constant along the paths of motion,

$$\left.\frac{df}{dt}\right|_{x^i(t)} = 0$$

**Definition:** Symmetry of the action Sometimes it is the case that  $\delta S$  vanishes for certain limited variations of the path without imposing any condition at all. When this happens, we say that S has a symmetry:

A symmetry of an action functional S[x] is a transformation of the path,  $x^{i}(t) \rightarrow \lambda^{i}(x^{j}(t), t)$  that leaves the action invariant,

$$S\left[x^{i}\left(t\right)\right] = S\left[\lambda^{i}\left(x^{j}\left(t\right),t\right)\right]$$

regardless of the path of motion  $x^{i}(t)$ . In particular, when  $\lambda^{i}(x)$  represents a continuous transformation of  $x^{1}$ , we may expand the transformation infinitesimally, so that

$$\begin{array}{rcl} x^{i} & \rightarrow & x'^{i} = x^{i} + \varepsilon^{i} \left( x \right) \\ \delta x^{i} & = & x'^{i} - x^{i} = \varepsilon^{i} \left( x \right) \end{array}$$

Since the infinitesimal transformation must leave S[x] invariant, we have

$$\delta_{\varepsilon}S = S\left[x^{i} + \varepsilon^{i}\left(x\right)\right] - S\left[x^{i}\right] = 0$$

whether x(t) satisfied the field equations or not. If an infinitesimal transformation is a symmetry, we may apply arbitrarily many infinitesimal transformations to recover the invariance of S under finite transformations. Here  $\lambda(x)$  is a particular function of the coordinates. This is quite different from performing a general variation – we are not placing any new demand on the action, just noticing that particular transformations don't change it. Notice that neither  $\lambda^i$  nor  $\varepsilon^i$  is required to vanish at the endpoints of the motion.

We are now in a position to prove Noether's theorem.

**Theorem (Noether):** Suppose an action dependent on N independent functions  $x^{i}(t)$ , i = 1, 2, ..., N has a (Lie) symmetry so that it is invariant under

$$\delta_{\varepsilon}x^{i} = x^{\prime i} - x^{i} = \varepsilon^{i}\left(x\right)$$

where  $\varepsilon^{i}(x)$  are fixed functions of  $x^{i}(t)$ . We carefully distinguish between the symmetry variation  $\delta_{\varepsilon}$  and a general variation  $\delta$ . Then the quantity

$$I = \frac{\partial L\left(x\left(\lambda\right)\right)}{\partial \dot{x}^{i}} \varepsilon^{i}\left(x\right)$$

is conserved.

**Proof:** The existence of a symmetry means that

$$\begin{array}{ll} 0 &\equiv& \delta_{\varepsilon}S\left[x\left(t\right)\right] \\ &\equiv& \sum_{i=1}^{N}\int_{t_{1}}^{t_{2}}\left(\frac{\partial L\left(x\left(t\right)\right)}{\partial x^{i}}\varepsilon^{i}\left(x\right) + \left(\frac{\partial L\left(x\left(t\right)\right)}{\partial \dot{x}_{\left(n\right)}^{i}}\right)\frac{d\varepsilon^{i}\left(x\right)}{dt}\right)dt \end{array}$$

Notice that  $\delta S$  vanishes identically because the action has a symmetry. No equation of motion has been used. Integrating the second term by parts we have

$$0 = \int \left(\frac{\partial L}{\partial x^{i}}\varepsilon^{i}(x) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{i}}\varepsilon^{i}(x)\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{i}}\right)\varepsilon^{i}(x)\right) dt$$
$$= \frac{\partial L}{\partial \dot{x}^{i}}\varepsilon^{i}(x)\Big|_{t_{1}}^{t_{2}} + \int \left(\frac{\partial L}{\partial x^{i}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{i}}\right)\right)\varepsilon^{i}(x)dt$$

This expression vanishes for *every* path. Now suppose  $x^{i}(t)$  is an actual classical path of the motion, that is, one that satisfies the Euler-Lagrange equation,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = 0$$

<sup>&</sup>lt;sup>1</sup>Technically, what we mean here is a Lie group of transformations, but the definition of a group lines up well with our intuition of symmetry. Groups are sets closed under an operation which has an identity, inverses and is associative. For symmetries, each transformation leaves the action invariant, so the combination of any two does as well, showing closure. The identity is just no transformation at all, inverses are just undoing the transformation we've just done, and associativity is natural if you can picture it – compounding three transformations ABC it doesn't matter whether we find AB and then apply first C then AB, or if we find BC and apply it, followed by A. It just means the symmetry transformation ABC is well-defined no matter which way we compute it.

Then for that classical path, the integrand vanishes and it follows that

$$\begin{array}{lll} 0 & = & \delta S\left[\mathbf{x}\right] \\ & = & \left. \frac{\partial L}{\partial \dot{x}^{i}} \varepsilon^{i}\left(x\left(t\right)\right) \right|_{t_{1}}^{t_{2}} \\ & = & I\left(t_{2}\right) - I\left(t_{1}\right) \end{array}$$

for any two end times,  $t_1, t_2$ . Therefore,

$$\frac{dI}{dt} = 0$$

and

$$I = \frac{\partial L\left(\mathbf{x}, \dot{\mathbf{x}}\right)}{\partial \dot{x}^{i}} \varepsilon^{i}\left(x\right)$$

is a constant of the motion.

## 2 Conserved quantities in Euler-Lagrange systems

We begin this section with some definitions.

Def: Cyclic coordinate A coordinate, q, is cyclic if it does not occur in the Lagrangian, i.e.,

$$\frac{\partial L}{\partial q} = 0$$

For example, in the spherically symmetric action

$$S[r,\theta,\varphi] = \int_{t_1}^{t_2} \left[ \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) - V(r) \right] dt$$

all three velocities  $(\dot{r}, \dot{\theta}, \dot{\varphi})$  are present and the coordinates  $(r, \theta)$  are present, but  $\frac{\partial L}{\partial \varphi} = 0$ . Therefore,  $\varphi$  is cyclic.

**Def:** Conjugate momentum The *conjugate momentum*, p, to any coordinate q is defined to be

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

For a single particle in any coordinate-dependent potential,  $V(\mathbf{x})$ , the action may be written as

$$S\left[\mathbf{x}\right] = \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - V\left(\mathbf{x}\right)\right] dt$$

so the momenta conjugate to the three coordinates  $x^i$  are

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m \dot{x}_i$$

which is the familiar expression for the momentum of a particle.

Notice that velocity dependence of the potential changes this. We give the example of the Lorentz force law,

$$\mathbf{F} = q \left( \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B} \right)$$

showing at the same time that we may use the velocity-dependent potential

$$q\phi - q\dot{\mathbf{x}}\cdot\mathbf{A}$$

where  $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ . Check this:

$$\begin{split} S\left[\mathbf{x}\right] &= \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}} \cdot \mathbf{A}\right] dt \\ 0 &= \delta S\left[\mathbf{x}\right] &= \int_{t_1}^{t_2} \left[m\dot{\mathbf{x}} \cdot \delta \dot{\mathbf{x}} - q\sum_i \frac{\partial \phi}{\partial x^i} \delta x^i + q\delta \dot{\mathbf{x}} \cdot \mathbf{A} + q\dot{\mathbf{x}} \cdot \sum_i \frac{\partial \mathbf{A}}{\partial x^i} \delta x^i\right] dt \\ 0 &= \int_{t_1}^{t_2} \left[-m\ddot{\mathbf{x}} \cdot \delta \mathbf{x} - q\sum_i \frac{\partial \phi}{\partial x^i} \delta x^i - q\delta \mathbf{x} \cdot \dot{\mathbf{A}} + q\dot{\mathbf{x}} \cdot \sum_i \frac{\partial \mathbf{A}}{\partial x^i} \delta x^i\right] dt \\ 0 &= \sum_i \int_{t_1}^{t_2} \left[-m\ddot{\mathbf{x}}_i - q\frac{\partial \phi}{\partial x^i} - q\frac{dA_i}{dt} + q\dot{\mathbf{x}} \cdot \frac{\partial \mathbf{A}}{\partial x^i}\right] \delta x^i dt \\ 0 &= \sum_i \int_{t_1}^{t_2} \left[-m\ddot{x}_i - q\frac{\partial \phi}{\partial x^i} - q\frac{\partial A_i}{\partial t} - q\sum_j \frac{\partial A_i}{\partial x^j}\dot{x}^j + q\sum_j \dot{x}^j\frac{\partial A_j}{\partial x^i}\right] \delta x^i dt \\ 0 &= \sum_i \int_{t_1}^{t_2} \left[-m\ddot{x}_i - q\left(\frac{\partial \phi}{\partial x^i} + \frac{\partial A_i}{\partial t}\right) + q\sum_j \dot{x}^j\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)\right] \delta x^i dt \\ 0 &= \sum_i \int_{t_1}^{t_2} \left[-m\ddot{x}_i - q\left(\frac{\partial \phi}{\partial x^i} + \frac{\partial A_i}{\partial t}\right) + q\sum_j \dot{x}^j\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)\right] \delta x^i dt \end{split}$$

so that

$$m\ddot{x}_{i} = q \left( E_{i} + \sum_{j} \dot{x}^{j} \epsilon_{ijk} B^{k} \right)$$
$$m\ddot{\mathbf{x}} = q \left( \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B} \right)$$

In this case, differentiating

$$p_j = \frac{\partial}{\partial \dot{x}^i} \left[ \frac{1}{2}m \sum_j \dot{x}^j \dot{x}_j - q\phi + q \sum_j \dot{x}^j A_j \right]$$
$$= m \dot{x}_j + q A_j$$

so that the momentum conjugate to  $x^i$  is

$$\mathbf{p} = m\dot{\mathbf{x}} + q\mathbf{A}$$

# 2.1 Cyclic coordinates and conserved momentum

We have the following consequences of a cyclic coordinate.

**Theorem: Cyclic coordinates** If a coordinate q is cyclic then

1. The system has translational symmetry, since the action is invariant under the translation

$$q \to q + a$$

2. The momentum conjugate to q is conserved.

**Proof:** To prove the first result, simply notice that if

$$\frac{\partial L}{\partial q} = 0$$

then L has no dependence on q at all. Therefore, replacing q by q + a does nothing to L, hence nothing to the action. Equivalently, the variation of the action with respect to the infinitesimal symmetry  $(a \rightarrow \varepsilon)$ ,

$$\begin{array}{rcl} \delta q & = & \varepsilon \\ \delta \dot{q} & = & 0 \end{array}$$

is

$$\delta S = \int \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right) dt$$
$$= \int \left(0 \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot 0\right) dt$$
$$= 0$$

so the translation is a symmetry of the action.

For the second result, the Euler-Lagrange equation for the coordinate q immediately gives

$$0 = \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)$$
$$= -\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)$$

so that

$$p = \frac{\partial L}{\partial \dot{q}}$$

is conserved.

#### 2.2 Translational invariance and conservation of momentum

Now consider full translational invariance. We look first at a single particle, then at many particles. Suppose the action for a 1-particle system is invariant under arbitrary finite translations,

$$\tilde{x}^i = x^i + a^i$$

or infinitesimally, letting  $a^i \to \varepsilon^i$ ,

$$\delta x^i = \varepsilon^i$$

We may express the invariance of S under  $\delta x^i = \varepsilon^i$  explicitly,

$$0 = \delta S$$

$$= \sum_{i} \int_{t_{1}}^{t_{2}} \left( \frac{\partial L}{\partial x^{i}} \delta x^{i} + \frac{\partial L}{\partial \dot{x}^{i}} \delta \dot{x}^{i} \right) dt$$

$$= \sum_{i} \int_{t_{1}}^{t_{2}} \left( \frac{\partial L}{\partial x^{i}} \delta x^{i} + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^{i}} \delta x^{i} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^{i}} \right) \delta x^{i} \right) dt$$

$$= \sum_{i} \frac{\partial L}{\partial \dot{x}^{i}} \varepsilon^{i} \Big|_{t_{1}}^{t_{2}} + \sum_{i} \int_{t_{1}}^{t_{2}} \left( \frac{\partial L}{\partial x^{i}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^{i}} \right) \right) \varepsilon^{i} dt$$

For a particle which satisfies the Euler-Lagrange equation, the final integral vanishes. Then, since  $t_1$  and  $t_2$  are arbitrary we must have

$$\frac{\partial L}{\partial \dot{x}^i} \varepsilon^i = p_i \varepsilon^i$$

conserved for all constants  $\varepsilon^i$ . Since  $\varepsilon^i$  is arbitrary, the momentum  $p_i = \frac{\partial L}{\partial \dot{x}^i}$  conjugate to  $x^i$  is conserved as a result of translational invariance.

Now consider an isolated system, i.e., a bounded system with potentials depending only on the relavite positions,  $\mathbf{x}_a - \mathbf{x}_b$  of the N particles (a, b = 1, ..., N). We may write the action for this system as

$$S\left[\mathbf{x}\right] = \sum_{a=1}^{N} \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{\mathbf{x}}_a^2 - \sum_{b \neq a} V\left(\mathbf{x}_a - \mathbf{x}_b\right)\right) dt$$

Then shifting the entire system by the same vector **a**,

$$\tilde{\mathbf{x}}_a = \mathbf{x}_a - \mathbf{a}$$

leaves S invariant since

$$\begin{aligned} \tilde{\mathbf{x}}_a - \tilde{\mathbf{x}}_b &= (\mathbf{x}_a - \mathbf{a}) - (\mathbf{x}_b - \mathbf{a}) = \mathbf{x}_a - \mathbf{x}_b \\ \dot{\tilde{\mathbf{x}}}_a &= \dot{\mathbf{x}}_a \end{aligned}$$

Carrying out the symmetry variation of S,

$$0 = \delta S[\mathbf{x}]$$

$$= \sum_{a=1}^{N} \int_{t_{1}}^{t_{2}} \left[ m \dot{\mathbf{x}}_{a} \cdot \delta \dot{\mathbf{x}}_{a} - \sum_{b \neq a} \sum_{c,i} \frac{\partial}{\partial x_{c}^{i}} V(\mathbf{x}_{a} - \mathbf{x}_{b}) a^{i} \right] dt$$

$$= \sum_{a=1}^{N} m \dot{\mathbf{x}}_{a} \cdot \delta \mathbf{x}_{a} + \sum_{a=1}^{N} \int_{t_{1}}^{t_{2}} \left[ m \ddot{\mathbf{x}}_{a} \cdot \delta \mathbf{x}_{a} - \sum_{b \neq a} \sum_{c} \delta \mathbf{x}_{c} \cdot \nabla_{c} V(\mathbf{x}_{a} - \mathbf{x}_{b}) \right] dt$$

$$= \sum_{a=1}^{N} m \dot{\mathbf{x}}_{a} \cdot \mathbf{a} + \sum_{a=1}^{N} \int_{t_{1}}^{t_{2}} \left[ m \ddot{\mathbf{x}}_{a} - \sum_{b \neq a} \sum_{c} \nabla_{c} V(\mathbf{x}_{a} - \mathbf{x}_{b}) \right] \cdot \mathbf{a} dt$$

Imposing the equations of motion, the final integral vanishes and the constant vector  $\mathbf{a}$  comes out of the initial sum so that

$$\mathbf{a} \cdot \sum_{a=1}^{N} m \dot{\mathbf{x}}_a = constant$$

Since  ${\bf a}$  is arbitrary, the total momentum

$$\mathbf{P} \equiv \sum_{a=1}^{N} m \dot{\mathbf{x}}_{a}$$

is conserved for an isolated system.