1   Why study Classical Mechanics?

Newton's laws date back to 1687, when he published Philosophiae Naturalis Principia Mathematica, laying out his laws of motion and universal gravitation. Now, over 300 years later, we understand the world in terms of relativistic quantum field theory or even fundamental string. Classical mechanics is known to fail in at least three ways. At distances smaller than $\frac{\hbar}{mv}$, we must use quantum mechanics or quantum field theory to describe the motions and interactions of matter. Second, at velocities close to the speed of light we must make relativistic corrections, working in a unified spacetime instead of Newton's abstraction of ideal Euclidan space and universal time. Finally, the law of universal gravitation has been replaced by general relativity. Why then, study classical mechanics at all?

1.1   Range of applicability

One answer lies in the wide range of applicability. Planck's constant $\hbar$ is small, so quantum considerations usually only become important for phenomena on the scale of atoms whose size is determined by $\frac{\hbar}{mv}$ for the orbiting electrons. Similarly, special relativity gives significant corrections only when objects move at a substantial fraction of the speed of light. The fastest man-made object ever produced was not the Voyager spacecraft (35,000 mi/hr) as is often claimed, but the solar probes Helios-A and Helios-B which reached a maximum speed of 252,792 km/hr (see http://en.wikipedia.org/wiki/Helios_(spacecraft)). This is still only $0.000234c$, with c the speed of light, so the relativistic corrections $\left(\sim \frac{v^2}{c^2} \right)$ are only a few parts in 10^8. Interestingly, second place appears to be held by a nuclear powered manhole cover (nuclear testing, Pascal B, gone wrong) (see http://savvyparanoia.com/the-fastest-man-made-object-ever-a-nuclear-powered-manhole-cover-true/) which would have been traveling at about 237,500 mph. This still is only $\frac{v}{c} = 2.2 \times 10^{-4}$. These small corrections are important only for extremely fine measurements, where they are easily measured by modern atomic clocks. Finally, general relativity is important for cosmology, precise orbit predictions in planetary and satellite science, and the GPS system, but errors using Newtonian gravity are of order $\frac{GM}{Rc^2}$, where R is the distane from a mass M. This is of order 6.95×10^{-7} near the surface of Earth.

   Therefore, for sizes larger than atoms and smaller than the solar system, and ordinary velocities, classical mechanics is an excellent approximation.

1.2   Mathematical techniques

A very important reason to study classical mechanics is that it provides a familiar, intuitive arena in which to learn solution techniques. For example, if we rewrite Newton's second law as the second-order differential equation,

$$\frac{d^2 x^i}{dt^2} = \frac{1}{m} F^i(x, v, t)$$

where $\left(x^i, F^i \right)$ for $i = 1, 2, 3$ are the components of the position and force vectors, it has the same form as the geodesic equation of general relativity,

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu$$

where $\alpha = 0, 1, 2, 3$ and the right side depends on $\Gamma^\alpha_{\mu\nu}\left(x^\beta \right)$ and the components of the velocity vector, u^μ.

   A second example is provided by the Schrödinger equation, which is identical in many respects to a classical diffusion equation.

1.3   First approximation

Because the corrections to Newtonian mechanics are so small, the Newtonian solution to problems is close to the exact solution, and therefore makes a good place to start in making a perturbative approximation to the full solution. Alternatively, if we have an exact solution in general relativity or quantum mechanics, we may be able to make sense of it by comparing terms in the classical solution.

1.4   Intuition

We have a great deal of direct experience with the world, and the terms of classical mechanics line up well with this experience. We can use this familiarity to guess how a system will behave. With more precise theories, having a similar picture of what is going on becomes difficult.

2   Review of Newtonian Mechanics

Basic definitions

We define several important concepts. We picture the world as a 3-dimensional Euclidean space with points labeled by triples of numbers, often simply the Cartesian (x, y, z), but others as well. Events are parameterized by the passage of time, so a point particle is described by a curve

$$\begin{aligned} \mathbf{r}\left(t\right) & = \left(x\left(t\right), y\left(t\right), z\left(t\right)\right) \\ & = \left(r\left(t\right), \theta\left(t\right), \varphi\left(t\right)\right) \end{aligned}$$

in whatever coordinates we choose, with boldface denoting a vector. Notice that the position vector $\mathbf{r}\left(t\right)$ is a *dynamical variable*, not a coordinate. As t varies, $\mathbf{r}\left(t\right)$ traces out a curve in space.

The time-rate-of-change of the position vector is called the *velocity*,

$$\begin{aligned} \mathbf{v}\left(t\right) & = \frac{d\mathbf{r}\left(t\right)}{dt} \\ & = \dot{\mathbf{r}}\left(t\right) \end{aligned}$$

and is tangent to this curve. Notice that for simplicity we will sometimes denote the time derivative with a dot over the variable. The *acceleration* is the rate of change of velocity,

$$\begin{aligned} \mathbf{a}\left(t\right) & = \frac{d\mathbf{v}\left(t\right)}{dt} \\ & = \frac{d^2\mathbf{r}\left(t\right)}{dt^2} \\ & = \ddot{\mathbf{r}}\left(t\right) \end{aligned}$$

Particles are characterized by a constant called the *mass*, m, which reflects their resistance to change of velocity. Defining the *momentum* as the product

$$\mathbf{p} = m\mathbf{v}$$

we write Newton's second law as

$$\mathbf{F} = \frac{d\mathbf{p}\left(t\right)}{dt}$$

The force, $\mathbf{F}$, is to be taken intuitivelty and is determined by the particular problem. It is essentially that effort which produces a change of momentum. For example, if we stretch a spring it has the ability to move a mass attached to the end. Since this ability doubles if we double the stretch of the spring, we may write the force of a spring as proportional to the difference between the location of its endpoints,

$$\mathbf{F}_{spring} = -k\left(\mathbf{r}_2 - \mathbf{r}_2\right)$$

where the spring constant, k, characterizes the strength of the spring. This form of force is Hooke's Law.

There are many forces that have been identified:

$$
\begin{array}{cc}
0 & Zero\ force \\
-k\left(\mathbf{r}_2 - \mathbf{r}_2\right) & Hooke's\ law \\
q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) & Lorentz\ force \\
\mathbf{N} & Normal\ force \\
-\mu\mathbf{N} & Friction \\
-\frac{GMm}{r^2}\hat{\mathbf{r}} & Gravitation \\
-\frac{kQq}{r^2}\hat{\mathbf{r}} & Coulomb's\ law
\end{array}
$$

Once the forces on a particle have been identified, Newton's second law becomes an ordinary, second order differential equation

$$
\sum \mathbf{F} = \frac{d}{dt}\left(m\frac{d\mathbf{r}\left(t\right)}{dt}\right)
$$

where the sum is over all forces on the particle. This means that two initial conditions are required to give a unique solution to a problem. If the time starts at $t = t_0$, then the initial conditions may be taken as the position and velocity at t_0,

$$
\begin{aligned}
\mathbf{r}_0 &= \mathbf{r}\left(t_0\right) \\
\mathbf{v}_0 &= \mathbf{v}\left(t_0\right)
\end{aligned}
$$

Conservation laws

We now develop three conservation laws.

Conservation of linear momentum

If no force acts, $\mathbf{F} = 0$ and we have

$$
\frac{d\mathbf{p}\left(t\right)}{dt} = 0
$$

Integrating gives

$$
\mathbf{p}\left(t\right) = p_0 = m\mathbf{v}\left(t_0\right)
$$

The linear momentum is therefore constant; we say that $\mathbf{p}$ is *conserved*.

Conservation of angular momentum

We define the *angular momentum* of a particle at position $\mathbf{r}\left(t\right)$, relative to a fixed position $\mathbf{R}$, to be

$$
\mathbf{L} = \left(\mathbf{r}\left(t\right) - \mathbf{R}\right) \times \mathbf{p}
$$

and the *torque* about the same location to be

$$
\mathbf{N} = \left(\mathbf{r}\left(t\right) - \mathbf{R}\right) \times \mathbf{F}
$$

Then taking the cross product of the relative position, $\mathbf{r}\left(t\right) - \mathbf{R}$, with Newton's second law, we have

$$
\begin{aligned}
\left(\mathbf{r}\left(t\right) - \mathbf{R}\right) \times \mathbf{F} &= \left(\mathbf{r}\left(t\right) - \mathbf{R}\right) \times \frac{d\mathbf{p}\left(t\right)}{dt} \\
\mathbf{N} &= \frac{d}{dt}\left(\left(\mathbf{r}\left(t\right) - \mathbf{R}\right) \times \mathbf{p}\left(t\right)\right) - \left(\frac{d}{dt}\left(\mathbf{r}\left(t\right) - \mathbf{R}\right)\right) \times \mathbf{p}\left(t\right)
\end{aligned}
$$

where we use the product rule on the right. Since

$$\frac{d}{dt}\left(\mathbf{r}\left(t\right)-\mathbf{R}\right) = \frac{d\mathbf{r}\left(t\right)}{dt}-\frac{d\mathbf{R}}{dt}$$
$$= \mathbf{v}\left(t\right)-0$$

and with $\mathbf{p}\left(t\right)=m\mathbf{v}\left(t\right)$, the right side becomes

$$\frac{d}{dt}\left(\left(\mathbf{r}\left(t\right)-\mathbf{R}\right)\times\mathbf{p}\left(t\right)\right)-\left(\frac{d}{dt}\left(\mathbf{r}\left(t\right)-\mathbf{R}\right)\right)\times\mathbf{p}\left(t\right) = \frac{d\mathbf{L}}{dt}-\mathbf{v}\left(t\right)\times m\mathbf{v}\left(t\right)$$
$$= \frac{d\mathbf{L}}{dt}$$

since $\mathbf{v}\left(t\right)\times\mathbf{v}\left(t\right)=0$. We therefore have

$$\mathbf{N} = \frac{d\mathbf{L}}{dt}$$

It follows immediately that $\mathbf{L}$ is conserved if the torque vanishes.

Conservation of energy

Suppose the force on a particle is a function of particle position, $\mathbf{F}=\mathbf{F}\left(\mathbf{r}\right)$. Then we may integrate the second law by taking the dot product with the velocity:

$$\mathbf{F} = \frac{d\mathbf{p}\left(t\right)}{dt}$$
$$\mathbf{F}\cdot\mathbf{v} = \frac{d\mathbf{p}}{dt}\cdot\mathbf{v}$$
$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}$$
$$\mathbf{F}\cdot d\mathbf{r} = m\mathbf{v}\cdot d\mathbf{v}$$

Integrating from the initial values to the values at a general time, t, we have

$$\int_{\mathbf{r}_0}^{\mathbf{r}(t)}\mathbf{F}\cdot d\mathbf{r} = m\int_{\mathbf{v}_0}^{\mathbf{v}(t)}\mathbf{v}\cdot d\mathbf{v}$$
$$= \frac{1}{2}m\mathbf{v}^2-\frac{1}{2}m\mathbf{v}_0^2$$
$$= T-T_0$$

where we define the *kinetic energy*, $T\left(t\right)=\frac{1}{2}m\mathbf{v}^2$. In general, the integral on the left side depends on the path of integration. Such a path-dependent integral is *not* a function, but is called instead a *functional*. Along any path we define the *work* as

$$W_{12}=\int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{F}\cdot d\mathbf{r}$$

so that the work is equal to the change in kinetic energy,

$$W_{12}=T_2-T_1$$

This is the work-energy theorem.

Let $\mathbf{x}(\lambda)$ be any curve in 3-space, and take it as the path of integration. Then the work is a functional, denoted with square brackets, $W[\mathbf{x}(\lambda)]$, where

$$
\begin{aligned}
W[\mathbf{x}(\lambda)] &= \int_{\mathbf{x}(\lambda)} \mathbf{F} \cdot d\mathbf{x} \\
&= \int_{\mathbf{x}(\lambda)} \mathbf{F} \cdot \frac{d\mathbf{x}}{d\lambda} d\lambda
\end{aligned}
$$

Notice that any change in the curve may change the value of W; the value of W depends on the entire curve, not just on a single point.

Suppose the result of integration is independent of path. Then the value of the integral depends only on the endpoints, and we define the *potential*,

$$
V(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}
$$

Since the potential evaluated along any two paths, C_1, C_2 is the same, the difference of the integrals along these paths must vanish

$$
\begin{aligned}
0 &= \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \\
&= \oint_{C_2 - C_1} \mathbf{F} \cdot d\mathbf{r}
\end{aligned}
$$

where $C_2 - C_1$ is the combined, closed loop up C_2 and back down C_1. Applying Stoke's theorem, this becomes

$$
\begin{aligned}
0 &= \oint_{C_2 - C_1} \mathbf{F} \cdot d\mathbf{r} \\
&= \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, d^2 x
\end{aligned}
$$

where S is any 2-dimensional surface with the curve $C_2 - C_1$ as its boundary, and $\hat{\mathbf{n}}$ is the unit normal to S. Since the closed curve $C_2 - C_1$ is completely arbitrary, the surface S is as well. Tipping S in arbitrary directions means that $\hat{\mathbf{n}}$ is also arbitrary. Choosing S to be an infinitesmally small region near a point $\mathbf{x}$, we may approximate the surface integral arbitrarily well by

$$
[(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}](\mathbf{x}) \, \Delta S
$$

where ΔS is the nonzero, infinitesimal area. This must vanish, and since the normal may be any direction, the curl of the force must vanish,

$$
\nabla \times \mathbf{F} = 0
$$

Since the point $\mathbf{x}$ was arbitrary, the curl must vanish at every point. This is a sufficient condition for the existence of a potential for $\mathbf{F}$.

To show that it is also necessary, we invert the definition of V to write the force as the gradient of the potential,

$$
\mathbf{F} = -\nabla V(\mathbf{r})
$$

Then, whenever the force comes from a potential, its curl must vanish, since the curl of a gradient is always zero:

$$
\begin{aligned}
\nabla \times \mathbf{F} &= -\nabla \times \nabla V(\mathbf{r}) \\
&= 0
\end{aligned}
$$