Hamiltonian Mechanics

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1 Phase space

Phase space is a dynamical arena for classical mechanics in which the number of independent dynamical variables is doubled from n variables q_i, $i = 1, 2, \ldots, n$ to $2n$ by treating either the velocities or the momenta as independent variables. This has two important consequences.

First, the equations of motion become first order differential equations instead of second order, so that the initial conditions is enough to specify a unique point in phase space. The means that, unlike the configurations space treatment, there is a unique solution to the equtions of motion through each point. This permits some useful geometric techniques in the study of the system.

Second, as we shall see, the set of transformations that preserve the equations of motion is enlarged. In Lagrangian mechanics, we are free to use n general coordinates, q_i, for our description. In phase space, however, we have $2n$ coordinates. Even though transformations among these $2n$ coordinates are not completely arbitrary, there are far more allowed transformations. This large set of transformations allows us, in principal, to formulate a general solution to mechanical problems via the Hamilton-Jacobi equation.

1.1 Velocity phase space

While we will not be using velocity phase space here, it provides some motivation for our developments in the next Sections. The formal presentation of Hamiltonian dynamics begins in Section 1.3.

Suppose we have an action functional

$$S = \int L\left(q_i, \dot{q}_j, t\right) dt$$

dependent on n dynamical variables, $q_i(t)$, and their time derivatives. We might instead treat $L(q_i, u_j, t)$ as a function of $2n$ dynamical variables. Thus, instead of treating the the velocities as time derivatives of the position variables, $(q_i, \dot{q}_i)$ we introduce n velocities u_i and treat them as independent. Then the variations of the velocities δu_i are also independent, and we end up with $2n$ equations. We then include n constraints, restoring the relationship between q_i and $\dot{q}_i$,

$$S = \int \left[L\left(q_i, u_j, t\right) + \sum \lambda_i \left(\dot{q}_i - u_i\right)\right] dt$$

Variation of the original dynamical variables then results in

$$\begin{aligned} 0 &= \delta_q S \\ &= \int \left(\frac{\partial L}{\partial q_i}\delta q_i + \lambda_i \delta \dot{q}_i\right) dt \\ &= \int \left(\frac{\partial L}{\partial q_i} - \dot{\lambda}_i\right) \delta q_i dt \end{aligned}$$

so that

$$\dot{\lambda}_i = \frac{\partial L}{\partial q_i}$$

For the velocities, we find

$$\begin{aligned} 0 &= \delta_u S \\ &= \int \left(\frac{\partial L}{\partial u_i} - \lambda_i \right) \delta u_i dt \end{aligned}$$

so that

$$\lambda_i = \frac{\partial L}{\partial u_i}$$

and finally, varying the Lagrange multipliers, λ_i, we recover the constraints,

$$u_i = \dot{q}_i$$

We may eliminate the multipliers by differentiating the velocity equation

$$\frac{d}{dt}(\lambda_i) = \frac{d}{dt}\left(\frac{\partial L}{\partial u_i} \right)$$

to find $\dot{\lambda}_i$, then substituting for u_i and $\dot{\lambda}_i$ into the q_i equation,

$$\begin{aligned} \dot{\lambda}_i &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) &= \frac{\partial L}{\partial q_i} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= 0 \end{aligned}$$

and we recover the Euler-Lagrange equations. If the kinetic energy is of the form $\sum_{i=1}^{n} \frac{1}{2} m u_i^2$, then the Lagrange multipliers are just the momenta,

$$\begin{aligned} \lambda_i &= \frac{\partial L}{\partial u_i} \\ &= m u_i \\ &= m \dot{q}_i \end{aligned}$$

1.2 Phase space

We can make the construction above more general by requiring the Lagrange multipliers to always be the conjugate momentum. Combining the constraint equation with the equation for λ_i we have

$$\lambda_i = \frac{\partial L}{\partial \dot{q}_i}$$

We now define the conjugate momentum to be exactly this derivative,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

Then the action becomes

$$\begin{aligned} S &= \int \left[L(q_i, u_j, t) + \sum p_i \dot{q}_i - \sum p_i u_i \right] dt \\ &= \int \left[L(q_i, u_j, t) - \sum p_i u_i + \sum p_i \dot{q}_i \right] dt \end{aligned}$$

For Lagrangians quadratic in the velocities, the first two terms become

$$\begin{aligned} L\left(q_i, u_j, t\right) - \sum p_i u_i &= L\left(q_i, \dot{q}, t\right) - \sum p_i \dot{q}_i \\ &= T - V - \sum p_i \dot{q}_i \\ &= -\left(T + V\right) \end{aligned}$$

We define this quantity to be the Hamiltonian,

$$H \equiv \sum p_i \dot{q}_i - L\left(q_i, \dot{q}, t\right)$$

Then

$$S = \int \left[\sum p_i \dot{q}_i - H\right] dt$$

This successfully eliminates the Lagrange multipliers from the formulation.

The term "phase space" is generally reserved for momentum phase space, spanned by coordinates q_i, p_j.

1.2.1 Legendre transformation

Notice that H is, by definition, independent of the velocities, since

$$\begin{aligned} H &= \sum p_j \dot{q}_j - L \\ \frac{\partial H}{\partial \dot{q}_i} &= \frac{\partial}{\partial \dot{q}_i}\left(\sum_j p_j \dot{q}_j - L\right) \\ &= \sum_j p_j \delta_{ij} - \frac{\partial L}{\partial \dot{q}_i} \\ &= p_i - \frac{\partial L}{\partial \dot{q}_i} \\ &\equiv 0 \end{aligned}$$

Therefore, the Hamiltonian is a function of q_i and p_i only. This is an example of a general technique called Legendre transformation. Suppose we have a function f, which depends on independent variables A, B and dependent variables, having partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial A} &= P \\ \frac{\partial f}{\partial B} &= Q \end{aligned}$$

Then the differential of f is

$$df = PdA + QdB$$

A Legendre transformation allows us to interchange variables to make either P or Q or both into the independent variables. For example, let $g\left(A, B, P\right) \equiv f - PA$. Then

$$\begin{aligned} dg &= df - AdP - PdA \\ &= PdA + QdB - AdP - PdA \\ &= QdB - AdP \end{aligned}$$

so that g actually only changes with B and P, $g = g(B, P)$. Similarly, $h = f - QB$ is a function of (A, Q) only, while $k = -(f - PA - QB)$ has (P, Q) as independent variables. Explicitly,

$$\begin{aligned} dk &= -df + PdA + AdP + QdB + BdQ \\ &= AdP + BdQ \end{aligned}$$

and we now have

$$\begin{aligned} \frac{\partial f}{\partial P} &= A \\ \frac{\partial f}{\partial Q} &= B \end{aligned}$$

2 Hamilton's equations

The essential formalism of Hamilton's equation is as follows. We begin with the action

$$S = \int L(q_i, \dot{q}_j, t)\, dt$$

and define the conjugate momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

and Hamiltonian

$$H(q_i, p_j, t) \equiv \sum p_j \dot{q}_j - L(q_i, \dot{q}_j, t)$$

Then the action may be written as

$$S = \int \left[\sum p_j \dot{q}_j - H(q_i, p_j, t)\right] dt$$

where q_i and p_j are now treated as independent variables.

Finding extrema of the action with respect to all $2n$ variables, we find:

$$\begin{aligned} 0 &= \delta_{q_k} S \\ &= \int \left(\left(\frac{\partial \sum_j p_j \dot{q}_j}{\partial \dot{q}_k} \right) \delta \dot{q}_k - \frac{\partial H}{\partial q_k} \delta q_k \right) dt \\ &= \int \left(\sum_j p_j \delta_{jk} \delta \dot{q}_k - \frac{\partial H}{\partial q_k} \delta q_k \right) dt \\ &= \int \left(p_k \delta \dot{q}_k - \frac{\partial H}{\partial q_k} \delta q_k \right) dt \\ &= \int \left(-\dot{p}_k - \frac{\partial H}{\partial q_k} \right) \delta q_k dt \end{aligned}$$

so that

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}$$

and

$$\begin{aligned} 0 &= \delta_{pk} S \\ &= \int \left(\left(\frac{\partial \sum_j p_j \dot{q}_j}{\partial p_k} \right) \delta p_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt \end{aligned}$$

$$
\begin{aligned}
&= \int \left(\dot{q}_k \delta p_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt \\
&= \int \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k dt
\end{aligned}
$$

so that

$$
\dot{q}_k = \frac{\partial H}{\partial p_k}
$$

These are Hamilton's equations. Whenever the Legendre transformation between L and H and between $\dot{q}_k$ and p_k is non-degenerate, Hamilton's equations,

$$
\begin{aligned}
\dot{q}_k &= \frac{\partial H}{\partial p_k} \\
\dot{p}_k &= -\frac{\partial H}{\partial q_k}
\end{aligned}
$$

form a system equivalent to the Euler-Lagrange or Newtonian equations.

2.1 Example: Newton's second law

Suppose the Lagrangian takes the form

$$
L = \frac{1}{2} m \dot{\mathbf{x}}^2 - V(\mathbf{x})
$$

Then the conjugate momenta are

$$
\begin{aligned}
p_i &= \frac{\partial L}{\partial \dot{x}_i} \\
&= m \dot{x}_i
\end{aligned}
$$

and the Hamiltonian becomes

$$
\begin{aligned}
H(x_i, p_j, t) &\equiv \sum p_j \dot{x}_j - L(x_i, \dot{x}_j, t) \\
&= m \dot{\mathbf{x}}^2 - \frac{1}{2} m \dot{\mathbf{x}}^2 + V(\mathbf{x}) \\
&= \frac{1}{2} m \dot{\mathbf{x}}^2 + V(\mathbf{x}) \\
&= \frac{1}{2m} \mathbf{p}^2 + V(\mathbf{x})
\end{aligned}
$$

Notice that we must invert the relationship between the momenta and the velocities,

$$
\dot{x}_i = \frac{p_i}{m}
$$

then expicitly replace all occurrences of the velocity with appropriate combinations of the momentum.

Hamilton's equations are:

$$
\begin{aligned}
\dot{x}_k &= \frac{\partial H}{\partial p_k} \\
&= \frac{p_k}{m} \\
\dot{p}_k &= -\frac{\partial H}{\partial x_k} \\
&= -\frac{\partial V}{\partial x_k}
\end{aligned}
$$

thereby reproducing the usual definition of momentum and Newton's second law.

2.2 Example

Suppose we have a coupled oscillator comprised of two identical pendula of length l and each of mass m, connected by a light spring with spring constant k. Then for small displacements, the action is

$$S=\int\left[\frac{1}{2}ml^2\left(\dot{\theta}_1^2+\dot{\theta}_2^2\right)-\frac{1}{2}k\left(l\sin\theta_1-l\sin\theta_2\right)^2-mgl\left(1-\cos\theta_1\right)-mgl\left(1-\cos\theta_2\right)\right]dt$$

which for small angles becomes approximately

$$S=\int\left[\frac{1}{2}ml^2\left(\dot{\theta}_1^2+\dot{\theta}_2^2\right)-\frac{1}{2}k\left(l\theta_1-l\theta_2\right)^2-\frac{1}{2}mgl\left(\theta_1^2+\theta_2^2\right)\right]dt$$

The conjugate momenta are:

$$\begin{aligned}
p_1 &= \frac{\partial L}{\partial\dot{\theta}_1}\\
&= ml^2\dot{\theta}_1\\
p_2 &= \frac{\partial L}{\partial\dot{\theta}_2}\\
&= ml^2\dot{\theta}_2
\end{aligned}$$

and the Hamiltonian is

$$\begin{aligned}
H &= p_1\dot{\theta}_1+p_2\dot{\theta}_2-L\\
&= ml^2\dot{\theta}_1^2+ml^2\dot{\theta}_2^2-\left(\frac{1}{2}ml^2\left(\dot{\theta}_1^2+\dot{\theta}_2^2\right)-\frac{1}{2}k\left(l\theta_1-l\theta_2\right)^2-\frac{1}{2}mgl\left(\theta_1^2+\theta_2^2\right)\right)\\
&= \frac{1}{2}ml^2\left(\dot{\theta}_1^2+\dot{\theta}_2^2\right)+\frac{1}{2}k\left(l\theta_1-l\theta_2\right)^2+\frac{1}{2}mgl\left(\theta_1^2+\theta_2^2\right)\\
&= \frac{1}{2ml^2}\left(p_1^2+p_2^2\right)+\frac{1}{2}k\left(l\theta_1-l\theta_2\right)^2+\frac{1}{2}mgl\left(\theta_1^2+\theta_2^2\right)
\end{aligned}$$

Notice that we always eliminate the velocities and write the Hamiltonian as a function of the momenta, p_i.

Hamilton's equations are:

$$\begin{aligned}
\dot{\theta}_1 &= \frac{\partial H}{\partial p_1}\\
&= \frac{1}{ml^2}p_1\\
\dot{\theta}_2 &= \frac{\partial H}{\partial p_2}\\
&= \frac{1}{ml^2}p_2\\
\dot{p}_1 &= -\frac{\partial H}{\partial\theta_1}\\
&= -\frac{1}{2}kl^2\left(\theta_1-\theta_2\right)-mgl\theta_1\\
\dot{p}_2 &= -\frac{\partial H}{\partial\theta_2}\\
&= \frac{1}{2}kl^2\left(\theta_1-\theta_2\right)-mgl\theta_2
\end{aligned}$$

From here we may solve in any way that suggests itself. If we differentiate $\dot{\theta}_1$ again, and use the third equation, we have

$$\begin{aligned}
\ddot{\theta}_1 &= \frac{1}{ml^2}\dot{p}_1 \\
&= -\frac{1}{ml^2}\frac{1}{2}kl^2(\theta_1-\theta_2) - \frac{g}{l}\theta_1 \\
&= -\frac{k}{m}\frac{1}{2}(\theta_1-\theta_2) - \frac{g}{l}\theta_1
\end{aligned}$$

Similarly, for θ_2 we have

$$\ddot{\theta}_2 = \frac{k}{m}\frac{1}{2}(\theta_1-\theta_2) - \frac{g}{l}\theta_2$$

Subtracting,

$$\begin{aligned}
\ddot{\theta}_1 - \ddot{\theta}_2 &= -\frac{k}{m}\frac{1}{2}(\theta_1-\theta_2) - \frac{k}{m}\frac{1}{2}(\theta_1-\theta_2) \\
\frac{d^2}{dt^2}(\theta_1-\theta_2) + \frac{k}{m}(\theta_1-\theta_2) &= 0
\end{aligned}$$

so that

$$\theta_1 - \theta_2 = A\sin\sqrt{\frac{k}{m}}t + B\cos\sqrt{\frac{k}{m}}t$$

Adding instead, we find

$$\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{l}(\theta_1+\theta_2)$$

so that

$$\theta_1 + \theta_2 = C\sin\sqrt{\frac{g}{l}}t + D\cos\sqrt{\frac{g}{l}}t$$

3 Formal developments

In order to fully appreciate the power and uses of Hamiltonian mechanics, we make some formal developments. First, we write Hamilton's equations,

$$\begin{aligned}
\dot{x}_k &= \frac{\partial H}{\partial p_k} \\
\dot{p}_k &= -\frac{\partial H}{\partial x_k}
\end{aligned}$$

for $k = 1, \ldots, n$, in a different way. Define a unified name for our $2n$ coordinates,

$$\xi_A = (x_i, p_j)$$

for $A = 1, \ldots, 2n$. That is, more explicitly,

$$\begin{aligned}
\xi_i &= x_i \\
\xi_{n+i} &= p_i
\end{aligned}$$

We may immediately write the left side of both of Hamilton's equations at once as

$$\dot{\xi}_A = (\dot{x}_i, \dot{p}_j)$$

The right side of the equations involves all of the terms

$$\frac{\partial H}{\partial \xi_A} = \left(\frac{\partial H}{\partial x_i}, \frac{\partial H}{\partial p_j} \right)$$

but there is a difference of a minus sign between the two equations and the interchange of x_i and p_i. We handle this by introducing a matrix called the *symplectic form*,

$$\Omega_{AB} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

where

$$[\mathbf{1}]_{ij} = \delta_{ij}$$

is the $n \times n$ identity matrix. Then, using the summation convention, Hamilton's equations take the form of a single expression,

$$\dot{\xi}_A = \Omega_{AB} \frac{\partial H}{\partial \xi_B}$$

We may check this by writing it out explicitly,

$$\begin{aligned} \begin{pmatrix} \dot{x}_i \\ \dot{p}_{i'} \end{pmatrix} &= \begin{pmatrix} 0 & \delta_{ij'} \\ -\delta_{i'j} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x_j} \\ \frac{\partial H}{\partial p_{j'}} \end{pmatrix} \\ &= \begin{pmatrix} \delta_{ij'} \frac{\partial H}{\partial p_{j'}} \\ -\delta_{i'j} \frac{\partial H}{\partial x_j} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial x_{i'}} \end{pmatrix} \end{aligned}$$

In the example above, we have $\xi_1 = \theta_1, \xi_2 = \theta_2, \xi_3 = p_1$ and $\xi_4 = p_2$. In terms of these, the Hamiltonian may be written as

$$\begin{aligned} H &= \frac{1}{2} H_{AB} \xi_A \xi_B \\ H_{AB} &= \begin{pmatrix} kl^2 + mgl & -kl^2 & 0 & 0 \\ -kl^2 & kl^2 + mgl & 0 & 0 \\ 0 & 0 & \frac{1}{ml^2} & 0 \\ 0 & 0 & 0 & \frac{1}{ml^2} \end{pmatrix} \end{aligned}$$

and with

$$\frac{\partial}{\partial \xi_C} H = \frac{1}{2} H_{AB} \delta_{AC} \xi_B + \frac{1}{2} H_{AB} \xi_A \delta_{BC} = H_{CB} \xi_B$$

Hamilton's equations are

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} kl^2 + mgl & -kl^2 & 0 & 0 \\ -kl^2 & kl^2 + mgl & 0 & 0 \\ 0 & 0 & \frac{1}{ml^2} & 0 \\ 0 & 0 & 0 & \frac{1}{ml^2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \frac{1}{ml^2} & 0 \\ 0 & 0 & 0 & \frac{1}{ml^2} \\ -kl^2 - mgl & kl^2 & 0 & 0 \\ kl^2 & -kl^2 - mgl & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{ml^2} \xi_3 \\ \frac{1}{ml^2} \xi_4 \\ -kl^2 \xi_1 - mgl \xi_1 + kl^2 \xi_2 \\ -kl^2 \xi_2 - mgl \xi_2 + kl^2 \xi_1 \end{pmatrix} \end{aligned}$$

so that

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{ml^2}\xi_3 \\ \frac{1}{ml^2}\xi_4 \\ -kl^2(\xi_1 - \xi_2) - mgl\xi_1 \\ kl^2(\xi_1 - \xi_2) - mgl\xi_2 \end{pmatrix}$$

as expected.

3.1 Properties of the symplectic form

We note a number of important properties of the symplectic form. First, it is antisymmetric,

$$\begin{aligned} \Omega^t &= -\Omega \\ \Omega_{AB} &= -\Omega_{BA} \end{aligned}$$

and it squares to minus the $2n$-dimensional identity,

$$\begin{aligned} \Omega^2 &= -\mathbf{1} \\ \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} &= \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \\ &= -\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \end{aligned}$$

We also have

$$\Omega^t = \Omega^{-1}$$

since $\Omega^t = -\Omega$, and therefore $\Omega\Omega^t = \Omega(-\Omega) = -\Omega^2 = \mathbf{1}$. Since all components of Ω_{AB} are constant, it is also true that

$$\partial_A \Omega_{BC} = \frac{\partial}{\partial \xi_A}\Omega_{BC} = 0$$

This last condition does not hold in every basis, however.

The defining properties of the symplectic form, necessary and sufficient to guarantee that it has the properties we require for Hamiltonian mechanics are that it be a $2n \times 2n$ matrix satisfying two properties at each point of phase space:

1. $\Omega^2 = -\mathbf{1}$
2. $\partial_A \Omega_{BC} + \partial_B \Omega_{CA} + \partial_C \Omega_{AB} = 0$

The first of these is enough for there to exist a change of basis so that $\Omega_{AB} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ at any given point, while the vanishing combination of derivatives insures that this may be done at *every* point of phase space.

3.2 Conservation and cyclic coordinates

From the relationship between the Lagrangian and the Hamiltonian,

$$H = p_i \dot{x}_i - L$$

we see that

$$\frac{\partial H}{\partial x_i} = -\frac{\partial L}{\partial x_i}$$

If the coordinate x_i is cyclic, $\frac{\partial L}{\partial x_i} = 0$, then the corresponding Hamilton equation reads

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial x_i} \\ &= 0 \end{aligned}$$

and the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

is conserved, so the relationship between cyclic coordinates and conserved quantities still holds.

Since the Lagrangian is independent of p_i, depending only on x_i and $\dot{x}_i$, we also have a corresponding statement about momentum. Suppose some momentum, p_i, is cyclic in the Hamiltonian,

$$\frac{\partial H}{\partial p_i} = 0$$

Then from either Hamilton's equations or from the relationship between the Hamiltonian and the Lagrangian, we immediately have

$$\dot{x}_i = 0$$

so that the coordinate x_i is a constant of the motion.

Suppose we have a cyclic coordinate, say x_n. Then the conserved momentum takes its initial value, p_{n0}, and the Hamiltonian is

$$H = H\left(x_1, \ldots x_{n-1}; p_1, \ldots p_{n-1}, p_{n0}\right)$$

and therefore immediately becomes a function of $n-1$ variables. This is simpler than the Lagrangian case, where constancy of p_n makes no immediate simplification of the Lagrangian.

Example 1: As a simple example, consider the 2-dimensional Kepler problem, with Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GM}{r}$$

The coordinate θ is cyclic and therefore

$$l = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

is conserved, but this quantity does not explicitly occur in the Lagrangian. However, the Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{GM}{r}$$

so $p_\theta = l$ is constant and we immediately have

$$H = \frac{1}{2m}\left(p_r^2 + \frac{l^2}{r^2}\right) - \frac{GM}{r}$$

Example 2 (problem 21a): Consider a flywheel of mass M and radius a, its center fixed . A rod of length a is attached to the perimeter with its other end constrained to lie on a horizontal line through the center of the flywheel, and is attached to the massless rod of a simple pendulum of length l and mass m. Find the Hamiltonian.

First, find L,

$$T = \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}I\dot{\varphi}^2$$

where $\dot{\varphi}(t)$ is the angular velocity of the flywheel and

$$\begin{aligned}
I_{33} &= \frac{M}{\pi a^2 d} \int \left(x^2 + y^2 + z^2 - z^2\right) dxdydz \\
&= \frac{M}{\pi a^2} \int \left(x^2 + y^2\right) dxdy \\
&= \frac{M}{\pi a^2} \int \rho^3 d\rho d\varphi \\
&= \frac{2\pi M}{\pi a^2} \frac{a^4}{4} \\
&= \frac{1}{2} M a^2
\end{aligned}$$

while the velocity of the pendulum bob is the combination of the swinging, $l\dot{\theta}$, and the oscillatory motion of the suspension point, located at $2a\cos\omega t$. With the postion of m given by

$$\begin{aligned}
x &= 2a\cos\varphi + l\sin\theta \\
y &= l\cos\theta
\end{aligned}$$

the velocity has components

$$\begin{aligned}
\dot{x} &= -2a\dot{\varphi}\sin\varphi + l\dot{\theta}\cos\theta \\
\dot{y} &= -l\dot{\theta}\sin\theta
\end{aligned}$$

Therefore,

$$\begin{aligned}
T &= \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}I\dot{\varphi}^2 \\
&= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{4}Ma^2\dot{\varphi}^2 \\
&= \frac{1}{2}m\left(4a^2\dot{\varphi}^2\sin^2\varphi - 4al\dot{\varphi}\dot{\theta}\sin\varphi\cos\theta + l^2\dot{\theta}^2\right) + \frac{1}{4}Ma^2\dot{\varphi}^2
\end{aligned}$$

and the potential is simply

$$V = -mgl\cos\theta$$

up to an arbitrary constant. Therefore,

$$L = \frac{1}{2}m\left(4a^2\dot{\varphi}^2\sin^2\varphi - 4al\dot{\varphi}\dot{\theta}\sin\varphi\cos\theta + l^2\dot{\theta}^2\right) + \frac{1}{4}Ma^2\dot{\varphi}^2 + mgl\cos\theta$$

To find the Hamiltonian, we first find the conjugate momenta,

$$\begin{aligned}
p_\theta &= \frac{\partial L}{\partial\dot{\theta}} \\
&= ml^2\dot{\theta} - 2mal\dot{\varphi}\sin\varphi\cos\theta \\
p_\varphi &= \frac{\partial L}{\partial\dot{\varphi}} \\
&= 4ma^2\dot{\varphi}\sin^2\varphi - 2mal\dot{\theta}\sin\varphi\cos\theta + \frac{1}{2}Ma^2\dot{\varphi}
\end{aligned}$$

Next, find the Hamiltonian in terms of both velocities and momenta,

$$\begin{aligned}
H &= \left(ml^2\dot{\theta} - 2mal\dot{\varphi}\sin\varphi\right)\dot{\theta} + \left(4ma^2\dot{\varphi}\sin^2\varphi - 2mal\dot{\theta}\sin\varphi + \frac{1}{2}Ma^2\dot{\varphi}\right)\dot{\varphi} \\
&\quad -\frac{1}{2}m\left(4a^2\dot{\varphi}^2\sin^2\varphi - 4al\dot{\varphi}\dot{\theta}\sin\varphi + l^2\dot{\theta}^2\right) - \frac{1}{4}Ma^2\dot{\varphi}^2 - mgl\cos\theta \\
&= \frac{1}{2}ml^2\dot{\theta}^2 - 2mal\dot{\varphi}\dot{\theta}\sin\varphi + 2ma^2\dot{\varphi}^2\sin^2\varphi + \frac{1}{4}Ma^2\dot{\varphi}^2 - mgl\cos\theta
\end{aligned}$$

Finally, solve for the velocities and eliminate them from H,

$$\begin{aligned}
\dot{\theta} &= \frac{p_\theta}{ml^2} + \frac{2a\dot{\varphi}}{l}\sin\varphi \\
p_\varphi + 2mal\dot{\theta}\sin\varphi &= 4ma^2\dot{\varphi}\sin^2\varphi + \frac{1}{2}Ma^2\dot{\varphi} \\
p_\varphi + 2mal\left(\frac{p_\theta}{ml^2} + \frac{2a\dot{\varphi}}{l}\sin\varphi\right)\sin\varphi &= 4ma^2\dot{\varphi}\sin^2\varphi + \frac{1}{2}Ma^2\dot{\varphi} \\
p_\varphi + \frac{2ap_\theta}{l}\sin\varphi &= 4ma^2\dot{\varphi}\sin^2\varphi + \frac{1}{2}Ma^2\dot{\varphi} - 4ma^2\dot{\varphi}\sin^2\varphi \\
&= \dot{\varphi}\left(4ma^2\sin^2\varphi + \frac{1}{2}Ma^2 - 4ma^2\sin^2\varphi\right) \\
&= \frac{1}{2}Ma^2\dot{\varphi} \\
\dot{\varphi} &= \frac{2}{Ma^2}p_\varphi + \frac{4\sin\varphi}{alM}p_\theta
\end{aligned}$$

and then

$$\begin{aligned}
\dot{\theta} &= \frac{p_\theta}{ml^2} + \frac{2a\dot{\varphi}}{l}\sin\varphi \\
&= \frac{p_\theta}{ml^2} + \frac{4}{Mal}p_\varphi\sin\varphi + \frac{8\sin^2\varphi}{l^2M}p_\theta \\
&= \frac{1}{ml^2}\left(1 + \frac{8m}{M}\sin^2\varphi\right)p_\theta + \frac{4}{Mal}p_\varphi\sin\varphi
\end{aligned}$$

so finally,

$$\begin{aligned}
H &= \frac{1}{2}ml^2\dot{\theta}^2 - 2mal\dot{\varphi}\dot{\theta}\sin\varphi + 2ma^2\dot{\varphi}^2\sin^2\varphi + \frac{1}{4}Ma^2\dot{\varphi}^2 - mgl\cos\theta \\
&= \frac{1}{2}ml^2\left(\frac{1}{(ml^2)^2}\left(1 + \frac{8m}{M}\sin^2\varphi\right)^2 p_\theta^2 + \frac{8\sin\varphi}{Mmal^3}\left(1 + \frac{8m}{M}\sin^2\varphi\right)p_\varphi p_\theta + \frac{16}{M^2a^2l^2}p_\varphi^2\sin^2\varphi\right) \\
&\quad -2mal\left(\frac{2}{Ma^2}p_\varphi + \frac{4\sin\varphi}{alM}p_\theta\right)\left(\frac{1}{ml^2}\left(1 + \frac{8m}{M}\sin^2\varphi\right)p_\theta + \frac{4}{Mal}p_\varphi\sin\varphi\right)\sin\varphi \\
&\quad + \left(2ma^2\sin^2\varphi + \frac{1}{4}Ma^2\right)\left(\frac{4}{M^2a^4}p_\varphi^2 + 2\frac{2}{Ma^2}p_\varphi\frac{4\sin\varphi}{alM}p_\theta + \frac{16\sin^2\varphi}{a^2l^2M^2}p_\theta^2\right) - mgl\cos\theta \\
&= \frac{1}{2ml^2}\left(1 + \frac{8m}{M}\sin^2\varphi\right)^2 p_\theta^2 + \frac{4\sin\varphi}{Mal}\left(1 + \frac{8m}{M}\sin^2\varphi\right)p_\varphi p_\theta + \frac{8m}{M^2a^2}p_\varphi^2\sin^2\varphi \\
&\quad -\frac{4\sin\varphi}{Mla}\left(1 + \frac{8m}{M}\sin^2\varphi\right)p_\varphi p_\theta - \frac{8\sin^2\varphi}{Ml^2}\left(1 + \frac{8m}{M}\sin^2\varphi\right)p_\theta p_\theta - \frac{8m}{M^2a^2}p_\varphi p_\varphi\sin^2\varphi - \frac{32m}{M^2al}p_\theta p_\varphi\sin^3\varphi \\
&\quad + \left(\frac{8m}{M^2a^2}\sin^2\varphi + \frac{1}{Ma^2}\right)p_\varphi^2 + \left(\frac{32m\sin^3\varphi}{alM^2} + \frac{4\sin\varphi}{alM}\right)p_\varphi p_\theta + \left(\frac{32m\sin^4\varphi}{l^2M^2} + \frac{4\sin^2\varphi}{l^2M}\right)p_\theta^2 - mgl\cos\theta
\end{aligned}$$

4 Note

My computer has swallowed the next ten pages of notes. Instead of rewriting it all now, I am copying relevant sections of my Mechanics book. I'll try to fill in any missing details.

5 Phase space and the symplectic form

We now explore some of the properties of phase space and Hamilton's equations.

One advantage of the Hamiltonian formulation is that there is now one equation for each initial condition. This gives the space of all *qs* and *ps* a uniqueness property that configuration space (the space spanned by the *qs* only) doesn't have. For example, a projectile which is launched from the origin. Knowing only this fact, we still don't know the path of the object – we need the initial velocity as well. As a result, many possible trajectories pass through each point of configuration space. By contrast, the initial point of a trajectory in phase space gives us *both* the initial position and the initial momentum. There can be only one path of the system that passes through that point.

Systems with any number of degrees of freedom may be handled in this way. If a system has N degrees of freedom then its phase space is the $2N$-dimensional space of all possible values of both position and momentum. We define *configuration space* to be the space of all possible postions of the particles comprising the system, or the complete set of possible values of the degrees of freedom of the problem. Thus, configuration space is the N-dimensional space of all values of q_i. By *momentum space*, we mean the N-dimensional space of all possible values of all of the conjugate momenta. Hamilton's equations then consist of $2N$ first order differential equations for the motion in phase space.

We illustrate these points with the simple example of a one dimensional harmonic oscillator.

Let a mass, m, free to move in one direction, experience a Hooke's law restoring force, $F = -kx$. Solve Hamilton's equations and study the motion of system in phase space. The Lagrangian for this system is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \end{aligned}$$

The conjugate momentum is just

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

so the Hamiltonian is

$$\begin{aligned} H &= p\dot{x} - L \\ &= \frac{p^2}{m} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2 \end{aligned}$$

Hamilton's equations are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} &= -\frac{\partial H}{\partial x} = -kx \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} = 0 \end{aligned}$$

Note that Hamilton's equations are two first-order equations. From this point on the coupled linear equations

$$\begin{aligned} \dot{p} &= -kx \\ \dot{x} &= \frac{p}{m} \end{aligned}$$

may be solved in any of a variety of ways. Let's treat it as a matrix system,

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} & \frac{1}{m} \\ -k & \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} \tag{1}$$

The matrix $M = \begin{pmatrix} & -k \\ \frac{1}{m} & \end{pmatrix}$ has eigenvalues $\omega = \left(\sqrt{\frac{k}{m}}, -\sqrt{\frac{k}{m}}\right)$ and diagonalizes to

$$\begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix} = AMA^{-1}$$

where

$$\begin{aligned}
A &= \frac{1}{2i\sqrt{km}} \left(\begin{array}{cc} i\sqrt{km} & 1 \\ i\sqrt{km} & -1 \end{array} \right) \\
A^{-1} &= \left(\begin{array}{cc} -1 & -1 \\ -i\sqrt{km} & i\sqrt{km} \end{array} \right) \\
\omega &= \sqrt{\frac{k}{m}}
\end{aligned}$$

Therefore, multiplying eq.(1) on the left by A and inserting $1 = A^{-1}A$,

$$\frac{d}{dt} A \left(\begin{array}{c} x \\ p \end{array} \right) = A \left(\begin{array}{cc} & \frac{1}{m} \\ -k & \end{array} \right) A^{-1} A \left(\begin{array}{c} x \\ p \end{array} \right) \tag{2}$$

we get decoupled equations in the new variables:

$$\left(\begin{array}{c} a \\ a^{\dagger} \end{array} \right) = A \left(\begin{array}{c} q \\ p \end{array} \right) = \left(\begin{array}{c} \frac{1}{2}\left(x - \frac{ip}{\sqrt{km}} \right) \\ \frac{1}{2}\left(x + \frac{ip}{\sqrt{km}} \right) \end{array} \right) \tag{3}$$

The decoupled equations are

$$\frac{d}{dt} \left(\begin{array}{c} a \\ a^{\dagger} \end{array} \right) = \left(\begin{array}{cc} -i\omega & 0 \\ 0 & i\omega \end{array} \right) \left(\begin{array}{c} a \\ a^{\dagger} \end{array} \right) \tag{4}$$

or simply

$$\begin{aligned}
\dot{a} &= -i\omega a \\
\dot{a}^{\dagger} &= -i\omega a^{\dagger}
\end{aligned}$$

with solutions

$$\begin{aligned}
a &= a_0 e^{-i\omega t} \\
a^{\dagger} &= a_0^{\dagger} e^{i\omega t}
\end{aligned}$$

The solutions for x and p may be written as

$$\begin{aligned}
x &= x_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t \\
p &= -m\omega x_0 \sin \omega t + p_0 \cos \omega t
\end{aligned}$$

Notice that once we specify the initial point in phase space, (x_0, p_0), the entire solution is determined. This solution gives a parameterized curve in phase space. To see what curve it is, note that

$$\begin{aligned}
\frac{m^2\omega^2 x^2}{2mE} + \frac{p^2}{2mE} &= \frac{m^2\omega^2 x^2}{p_0^2 + m^2\omega^2 x_0^2} + \frac{p^2}{p_0^2 + m^2\omega^2 x_0^2} \\
&= \frac{m^2\omega^2}{p_0^2 + m^2\omega^2 x_0^2} \left(x_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t \right)^2 \\
&\quad + \frac{1}{p_0^2 + m^2\omega^2 x_0^2} \left(-m\omega x_0 \sin \omega t + p_0 \cos \omega t \right)^2 \\
&= \frac{m^2\omega^2 x_0^2}{p_0^2 + m^2\omega^2 x_0^2} + \frac{p_0^2}{p_0^2 + m^2\omega^2 x_0^2} \\
&= 1
\end{aligned}$$

or

$$m^2\omega^2x^2 + p^2 = 2mE$$

This describes an ellipse in the xp plane. The larger the energy, the larger the ellipse, so the possible motions of the system give a set of nested, non-intersecting ellipses. Clearly, every point of the xp plane lies on exactly one ellipse.

The phase space description of classical systems are equivalent to the configuration space solutions and are often easier to interpret because more information is displayed at once. The price we pay for this is the doubled dimension – paths rapidly become difficult to plot. To ofset this problem, we can use *Poincaré sections* – projections of the phase space plot onto subspaces that cut across the trajectories. Sometimes the patterns that occur on Poincaré sections show that the motion is confined to specific regions of phase space, even when the motion never repeats itself. These techniques allow us to study systems that are *chaotic*, meaning that the phase space paths through nearby points diverge rapidly.

Now consider the general case of N degrees of freedom. Let

$$\xi^A = \left(q^i, p_j\right)$$

where $A = 1, \ldots, 2N$. Then the $2N$ variables ξ^A provide a set of coordinates for phase space. We would like to write Hamilton's equations in terms of these, thereby treating all $2N$ directions on an equal footing.

In terms of ξ^A, we have

$$\begin{aligned} \frac{d\xi^A}{dt} &= \begin{pmatrix} \dot{q}^i \\ \dot{p}_j \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q^j} \end{pmatrix} \\ &= \Omega^{AB}\frac{\partial H}{\partial \xi^B} \end{aligned}$$

where the presence of Ω^{AB} in the last step takes care of the difference in signs on the right. Here Ω^{AB} is just the inverse of the symplectic form found from the curl of the dilatation, given by

$$\Omega^{AB} = \begin{pmatrix} 0 & \delta^i_j \\ -\delta^j_i & 0 \end{pmatrix}$$

Its occurrence in Hamilton's equations is an indication of its central importance in Hamiltonian mechanics. We may now write Hamilton's equations as

$$\frac{d\xi^A}{dt} = \Omega^{AB}\frac{\partial H}{\partial \xi^B} \tag{5}$$

Consider what happens to Hamilton's equations if we want to change to a new set of phase space coordinates, $\chi^A = \chi^A(\xi)$. Let the inverse transformation be $\xi^A(\chi)$. The time derivatives become

$$\frac{d\xi^A}{dt} = \frac{\partial \xi^A}{\partial \chi^B}\frac{d\chi^B}{dt}$$

while the right side becomes

$$\Omega^{AB}\frac{\partial H}{\partial \xi^B} = \Omega^{AB}\frac{\partial \chi^C}{\partial \xi^B}\frac{\partial H}{\partial \chi^C}$$

Equating these expressions,

$$\frac{\partial \xi^A}{\partial \chi^B}\frac{d\chi^B}{dt} = \Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D}$$

we multiply by the Jacobian matrix, $\frac{\partial \chi^C}{\partial \xi^A}$ to get

$$
\begin{aligned}
\frac{\partial \chi^C}{\partial \xi^A}\frac{\partial \xi^A}{\partial \chi^B}\frac{d\chi^B}{dt} &= \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D} \\
\delta^C_B\frac{d\chi^B}{dt} &= \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D}
\end{aligned}
$$

and finally

$$
\frac{d\chi^C}{dt} = \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}\frac{\partial H}{\partial \chi^D}
$$

Defining the symplectic form in the new coordinate system,

$$
\tilde{\Omega}^{CD} \equiv \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}
$$

we see that Hamilton's equations are entirely the same if the transformation leaves the symplectic form invariant,

$$
\tilde{\Omega}^{CD} = \Omega^{CD}
$$

Any linear transformation $M^A{}_B$ leaving the symplectic form invariant,

$$
\Omega^{AB} \equiv M^A{}_C M^B{}_D \Omega^{CD}
$$

is called a symplectic transformation. Coordinate transformations which are symplectic transformations at each point are called *canonical.* Therefore those functions $\chi^A(\xi)$ satisfying

$$
\Omega^{CD} \equiv \frac{\partial \chi^C}{\partial \xi^A}\Omega^{AB}\frac{\partial \chi^D}{\partial \xi^B}
$$

are canonical transformations. Canonical transformations preserve Hamilton's equations.

5.1 Poisson brackets

We may also write Hamilton's equations in terms of the Poisson brackets. Recall that the Poisson bracket of any two dynamical variables f and g is given by

$$
\{f, g\} = \Omega^{AB}\frac{\partial f}{\partial \xi^A}\frac{\partial g}{\partial \xi^B}
$$

The importance of this product is that it too is preserved by canonical transformations. We see this as follows.

Let ξ^A be any set of phase space coordinates in which Hamilton's equations take the form of eq.(5), and let f and g be any two dynamical variables, that is, functions of these phase space coordinates, ξ^A. The Poisson bracket of f and g is given above. In a different set of coordinates, $\chi^A(\xi)$, we have

$$
\begin{aligned}
\{f, g\}' &= \Omega^{AB}\frac{\partial f}{\partial \chi^A}\frac{\partial g}{\partial \chi^B} \\
&= \Omega^{AB}\left(\frac{\partial \xi^C}{\partial \chi^A}\frac{\partial f}{\partial \xi^C}\right)\left(\frac{\partial \xi^D}{\partial \chi^B}\frac{\partial g}{\partial \xi^D}\right) \\
&= \left(\frac{\partial \xi^C}{\partial \chi^A}\Omega^{AB}\frac{\partial \xi^D}{\partial \chi^B}\right)\frac{\partial f}{\partial \xi^C}\frac{\partial g}{\partial \xi^D}
\end{aligned}
$$

Therefore, if the coordinate transformation is canonical so that

$$
\frac{\partial \xi^C}{\partial \chi^A}\Omega^{AB}\frac{\partial \xi^D}{\partial \chi^B} = \Omega^{CD}
$$

then we have

$$\{f,g\}' = \Omega^{AB}\frac{\partial f}{\partial \xi^C}\frac{\partial g}{\partial \xi^D} = \{f,g\}$$

and the Poisson bracket is unchanged. We conclude that canonical transformations preserve all Poisson brackets.

An important special case of the Poisson bracket occurs when one of the functions is the Hamiltonian. In that case, we have

$$\begin{aligned}
\{f,H\} &= \Omega^{AB}\frac{\partial f}{\partial \xi^A}\frac{\partial H}{\partial \xi^B} \\
&= \frac{\partial f}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p^i}\frac{\partial H}{\partial x_i} \\
&= \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} - \frac{\partial f}{\partial p^i}\left(-\frac{dp_i}{dt}\right) \\
&= \frac{df}{\partial t} - \frac{\partial f}{\partial t}
\end{aligned}$$

or simply,

$$\frac{df}{\partial t} = \{f,H\} + \frac{\partial f}{\partial t}$$

This shows that as the system evolves classically, the total time rate of change of any dynamical variable is the sum of the Poisson bracket with the Hamiltonian and the partial time derivative. If a dynamical variable has no explicit time dependence, then $\frac{\partial f}{\partial t} = 0$ and the total time derivative is just the Poisson bracket with the Hamiltonian.

The coordinates now provide a special case. Since neither x^i nor p_i has any explicit time dependence, with have

$$\begin{aligned}
\frac{dx^i}{dt} &= \left\{H, x^i\right\} \\
\frac{dp_i}{dt} &= \{H, p_i\}
\end{aligned} \tag{6}$$

and we can check this directly:

$$\begin{aligned}
\frac{dq_i}{dt} &= \left\{H, x^i\right\} \\
&= \sum_{j=1}^{N}\left(\frac{\partial x^i}{\partial x^j}\frac{\partial H}{\partial p_j} - \frac{\partial x^i}{\partial p_j}\frac{\partial H}{\partial x^j}\right) \\
&= \sum_{j=1}^{N}\delta_{ij}\frac{\partial H}{\partial p_j} \\
&= \frac{\partial H}{\partial p_i}
\end{aligned}$$

and

$$\begin{aligned}
\frac{dp_i}{dt} &= \{H, p_i\} \\
&= \sum_{j=1}^{N}\left(\frac{\partial p_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q_j}\right) \\
&= -\frac{\partial H}{\partial q_i}
\end{aligned}$$

Notice that since q_i, p_i and are all independent, and do not depend explicitly on time, $\frac{\partial q_i}{\partial p_j} = \frac{\partial p_i}{\partial q_j} = 0 = \frac{\partial q_i}{\partial t} = \frac{\partial p_i}{\partial t}$.

Finally, we define the *fundamental Poisson brackets.* Suppose x^i and p_j are a set of coordinates on phase space such that Hamilton's equations hold in the either the form of eqs.(6) or of eqs.(5). Since they themselves are functions of (x^m, p_n) they are dynamical variables and we may compute their Poisson brackets with one another. With $\xi^A = (x^m, p_n)$ we have

$$\begin{aligned}
\left\{x^i, x^j\right\}_\xi &= \Omega^{AB} \frac{\partial x^i}{\partial \xi^A} \frac{\partial x^j}{\partial \xi^B} \\
&= \sum_{m=1}^{N} \left(\frac{\partial x^i}{\partial x^m} \frac{\partial x^j}{\partial p_m} - \frac{\partial x^i}{\partial p_m} \frac{\partial x^j}{\partial x^m} \right) \\
&= 0
\end{aligned}$$

for x^i with x^j,

$$\begin{aligned}
\left\{x^i, p_j\right\}_\xi &= \Omega^{AB} \frac{\partial x^i}{\partial \xi^A} \frac{\partial p_j}{\partial \xi^B} \\
&= \sum_{m=1}^{N} \left(\frac{\partial x^i}{\partial x^m} \frac{\partial p_j}{\partial p_m} - \frac{\partial x^i}{\partial p_m} \frac{\partial p_j}{\partial x^m} \right) \\
&= \sum_{m=1}^{N} \delta^i_m \delta^m_j \\
&= \delta^i_j
\end{aligned}$$

for x^i with p_j and finally

$$\begin{aligned}
\left\{p_i, p_j\right\}_\xi &= \Omega^{AB} \frac{\partial p_i}{\partial \xi^A} \frac{\partial p_j}{\partial \xi^B} \\
&= \sum_{m=1}^{N} \left(\frac{\partial p_i}{\partial x^m} \frac{\partial p_j}{\partial p_m} - \frac{\partial p_i}{\partial p_m} \frac{\partial p_j}{\partial x^m} \right) \\
&= 0
\end{aligned}$$

for p_i with p_j. The subscript ξ on the bracket indicates that the partial derivatives are taken with respect to the coordinates $\xi^A = \left(x^i, p_j\right)$. We summarize these relations as

$$\left\{\xi^A, \xi^B\right\}_\xi = \Omega^{AB}$$

We summarize the results of this subsection with a theorem: Let the coordinates ξ^A be canonical. Then a transformation $\chi^A(\xi)$ is canonical if and only if it satisfies the fundamental bracket relation

$$\left\{\chi^A, \chi^B\right\}_\xi = \Omega^{AB}$$

For proof, note that the bracket on the left is defined by

$$\left\{\chi^A, \chi^B\right\}_\xi = \Omega^{CD} \frac{\partial \chi^A}{\partial \xi^C} \frac{\partial \chi^B}{\partial \xi^D}$$

so in order for χ^A to satisfy the canonical bracket we must have

$$\Omega^{CD} \frac{\partial \chi^A}{\partial \xi^C} \frac{\partial \chi^B}{\partial \xi^D} = \Omega^{AB} \tag{7}$$

which is just the condition shown above for a coordinate transformation to be canonical. Conversely, suppose the transformation $\chi^A(\xi)$ is canonical and $\left\{\xi^A, \xi^B\right\}_\xi = \Omega^{AB}$. Then eq.(7) holds and we have

$$\left\{\chi^A, \chi^B\right\}_\xi = \Omega^{CD} \frac{\partial \chi^A}{\partial \xi^C} \frac{\partial \chi^B}{\partial \xi^D} = \Omega^{AB}$$

so χ^A satisfies the fundamental bracked relation.

In summary, each of the following statements is equivalent:

1. $\chi^A(\xi)$ is a canonical transformation.
2. $\chi^A(\xi)$ is a coordinate transformation of phase space that preserves Hamilton's equations.
3. $\chi^A(\xi)$ preserves the symplectic form, according to

$$\Omega^{AB} \frac{\partial \xi^C}{\partial \chi^A} \frac{\partial \xi^D}{\partial \chi^B} = \Omega^{CD}$$

4. $\chi^A(\xi)$ satisfies the fundamental bracket relations

$$\left\{\chi^A, \chi^B\right\}_\xi = \Omega^{CD} \frac{\partial \chi^A}{\partial \xi^C} \frac{\partial \chi^B}{\partial \xi^D}$$

These bracket relations represent a set of integrability conditions that must be satisfied by any new set of canonical coordinates. When we formulate the problem of canonical transformations in these terms, it is not obvious what functions $q^i\left(x^j, p_j\right)$ and $\pi_i\left(x^j, p_j\right)$ will be allowed. Fortunately there is a simple procedure for generating canonical transformations, which we develop in the next section.

We end this section with three examples of canonical transformations.

5.1.1 Example 1: Coordinate transformations

Let the new configuration space variable, q^i, be and an arbitrary function of the spatial coordinates:

$$q^i = q^i\left(x^j\right)$$

and let π_j be the momentum variables corresponding to q^i. Then $\left(q^i, \pi_j\right)$ satisfy the fundamental Poisson bracket relations iff:

$$\begin{aligned} \left\{q^i, q^j\right\}_{x,p} &= 0 \\ \left\{q^i, \pi_j\right\}_{x,p} &= \delta^i_j \\ \left\{\pi_i, \pi_j\right\}_{x,p} &= 0 \end{aligned}$$

Check each:

$$\begin{aligned} \left\{q^i, q^j\right\}_{x,p} &= \sum_{m=1}^{N} \left(\frac{\partial q^i}{\partial x^m} \frac{\partial q^j}{\partial p_m} - \frac{\partial q^i}{\partial p_m} \frac{\partial q^j}{\partial x^m} \right) \\ &= 0 \end{aligned}$$

since $\frac{\partial q^j}{\partial p_m} = 0$. For the second bracket,

$$\begin{aligned} \delta^i_j &= \left\{q^i, \pi_j\right\}_{x,p} \\ &= \sum_{m=1}^{N} \left(\frac{\partial q^i}{\partial x^m} \frac{\partial \pi_j}{\partial p_m} - \frac{\partial q^i}{\partial p_m} \frac{\partial \pi_j}{\partial x^m} \right) \\ &= \sum_{m=1}^{N} \frac{\partial q^i}{\partial x^m} \frac{\partial \pi_j}{\partial p_m} \end{aligned}$$

Since q^i is independent of p_m, we can satisfy this only if

$$\frac{\partial \pi_j}{\partial p_m} = \frac{\partial x^m}{\partial q^j}$$

Integrating gives

$$\pi_j = \frac{\partial x^n}{\partial q^j} p_n + c_j$$

with c_j an arbitrary constant. The presence of c_j does not affect the value of the Poisson bracket. Choosing $c_j = 0$, we compute the final bracket:

$$\begin{aligned}
\{\pi_i, \pi_j\}_{x,p} &= \sum_{m=1}^{N} \left(\frac{\partial \pi_i}{\partial x^m} \frac{\partial \pi_j}{\partial p_m} - \frac{\partial \pi_i}{\partial p_m} \frac{\partial \pi_j}{\partial x^m} \right) \\
&= \sum_{m=1}^{N} \left(\frac{\partial^2 x^n}{\partial x^m \partial q^i} p_n \frac{\partial x^m}{\partial q^j} - \frac{\partial x^m}{\partial q^i} \frac{\partial^2 x^n}{\partial x^m \partial q^j} p_n \right) \\
&= \sum_{m=1}^{N} \left(\frac{\partial x^m}{\partial q^j} \frac{\partial}{\partial x^m} \frac{\partial x^n}{\partial q^i} - \frac{\partial x^m}{\partial q^i} \frac{\partial}{\partial x^m} \frac{\partial x^n}{\partial q^j} \right) p_n \\
&= \sum_{m=1}^{N} \left(\frac{\partial}{\partial q^j} \frac{\partial x^n}{\partial q^i} - \frac{\partial}{\partial q^i} \frac{\partial x^n}{\partial q^j} \right) p_n \\
&= 0
\end{aligned}$$

Therefore, the transformations

$$\begin{aligned}
q^j &= q^j(x^i) \\
\pi_j &= \frac{\partial x^n}{\partial q^j} p_n + c_j
\end{aligned}$$

is a canonical transformation for any functions $q^i(x)$. This means that the symmetry group of Hamilton's equations is at least as big as the symmetry group of the Euler-Lagrange equations.

5.1.2 Example 2: Interchange of x and p.

The transformation

$$\begin{aligned}
q^i &= p_i \\
\pi_i &= -x^i
\end{aligned}$$

is canonical. We easily check the fundamental brackets:

$$\begin{aligned}
\left\{q^i, q^j\right\}_{x,p} &= \{p_i, p_j\}_{x,p} = 0 \\
\left\{q^i, \pi_j\right\}_{x,p} &= \left\{p_i, -x^j\right\}_{x,p} \\
&= -\left\{p_i, x^j\right\}_{x,p} \\
&= +\left\{x^j, p_i\right\}_{x,p} \\
&= \delta_i^j \\
\{\pi_i, \pi_j\}_{x,p} &= \left\{-x^i, -x^j\right\}_{x,p} = 0
\end{aligned}$$

Interchange of x^i and p_j, with a sign, is therefore canonical. The use of generalized coordinates does not include such a possibility, so Hamiltonian dynamics has a larger symmetry group than Lagrangian dynamics.

For our last example, we first show that the composition of two canonical transformations is also canonical. Let $\psi(\chi)$ and $\chi(\xi)$ both be canonical. Defining the composition transformation, $\psi(\xi) = \psi(\chi(\xi))$, we compute

$$\begin{aligned}
\Omega^{CD}\frac{\partial\psi^A}{\partial\xi^C}\frac{\partial\psi^B}{\partial\xi^D} &= \Omega^{CD}\left(\frac{\partial\psi^A}{\partial\chi^E}\frac{\partial\chi^E}{\partial\xi^C}\right)\left(\frac{\partial\psi^B}{\partial\chi^F}\frac{\partial\chi^F}{\partial\xi^D}\right) \\
&= \frac{\partial\chi^E}{\partial\xi^C}\frac{\partial\chi^F}{\partial\xi^D}\Omega^{CD}\left(\frac{\partial\psi^A}{\partial\chi^E}\right)\left(\frac{\partial\psi^B}{\partial\chi^F}\right) \\
&= \Omega^{EF}\left(\frac{\partial\psi^A}{\partial\chi^E}\right)\left(\frac{\partial\psi^B}{\partial\chi^F}\right) \\
&= \Omega^{AB}
\end{aligned}$$

so that $\psi(\chi(\xi))$ is canonical.

5.1.3 Example 3: Momentum transformations

By the previous results, the composition of an arbitratry coordinate change with x, p interchanges is canonical. Consider the effect of composing (a) an interchange, (b) a coordinate transformation, and (c) an interchange.

For (a), let

$$\begin{aligned}
q_1^i &= p_i \\
\pi_i^1 &= -x^i
\end{aligned}$$

Then for (b) we choose an arbitrary function of q_1^i :

$$\begin{aligned}
Q^i &= Q^i\left(q_1^j\right) = Q^i(p_j) \\
P_i &= \frac{\partial q_1^n}{\partial Q^i}\pi_n = -\frac{\partial p_n}{\partial Q^i}x^n
\end{aligned}$$

Finally, for (c), another interchange:

$$\begin{aligned}
q^i &= P_i = -\frac{\partial p_n}{\partial Q^i}x^n \\
\pi_i &= -Q^i = -Q^i(p_j)
\end{aligned}$$

This establishes that replacing the momenta by any three independent functions of the momenta, preserves Hamilton's equations.

5.2 Generating functions

There is a systematic approach to canonical transformations using generating functions. We will give a simple example of the technique. Given a system described by a Hamiltonian $H(x^i, p_j)$, we seek another Hamiltonian $H'(q^i, \pi_j)$ such that the equations of motion have the same form, namely

$$\begin{aligned}
\frac{dx^i}{dt} &= \frac{\partial H}{\partial p_i} \\
\frac{dp_i}{dt} &= -\frac{\partial H}{\partial x^i}
\end{aligned}$$

in the original system and

$$\begin{aligned}\frac{dq^i}{dt} &= \frac{\partial H'}{\partial \pi_i}\\ \frac{d\pi_i}{dt} &= -\frac{\partial H'}{\partial q^i}\end{aligned}$$

in the transformed variables. The principle of least action must hold for each pair:

$$\begin{aligned}S &= \int \left(p_i dx^i - H dt\right)\\ S' &= \int \left(\pi_i dq^i - H' dt\right)\end{aligned}$$

where S and S' differ by at most a constant. Correspondingly, the integrands may differ by the addition of a total differential, $df = \frac{df}{dt}dt$, since this will integrate to a surface term and therefore will not contribute to the variation. Notice that this corresponds exactly to a local dilatation, which produces a change

$$\begin{aligned}W'_\alpha dx^\alpha &= W_\alpha dx^\alpha - df\\ &= W_\alpha dx^\alpha - \frac{df}{dt}dt\end{aligned}$$

In general we may therefore write

$$p_i dx^i - H dt = \pi_i dq^i - H' dt + df$$

A convenient way to analyze the condition is to solve it for the differential df

$$df = p_i dx^i - \pi_i dq^i + (H' - H)\, dt$$

For the differential of f to take this form, it must be a function of x^i, q^i and t, that is, $f = f(x^i, q^i, t)$. Therefore, the differential of f is

$$df = \frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial q^i}dq^i + \frac{\partial f}{\partial t}dt$$

Equating the expressions for df we match up terms to require

$$p_i = \frac{\partial f}{\partial x^i} \tag{8}$$

$$\pi_i = -\frac{\partial f}{\partial q^i} \tag{9}$$

$$H' = H + \frac{\partial f}{\partial t} \tag{10}$$

The first equation

$$p_i = \frac{\partial f(x^j, q^j, t)}{\partial x^i} \tag{11}$$

gives q^i implicitly in terms of the original variables, while the second determines π_i. Notice that we may pick any function $q^i = q^i(p_j, x^j, t)$. This choice fixes the form of π_i by the eq.(9), while the eq.(10) gives the new Hamiltonian in terms of the old one. The function f is the generating function of the transformation.

6 General solution in Hamiltonian dynamics

We conclude with the crowning theorem of Hamiltonian dynamics: a proof that for any Hamiltonian dynamical system there exists a canonical transformation to a set of variables on phase space such that the paths of motion reduce to single points. Clearly, this theorem shows the power of canonical transformations! The theorem relies on describing solutions to the Hamilton-Jacobi equation, which we introduce first.

6.1 The Hamilton-Jacobi Equation

We have the following equations governing Hamilton's principal function.

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial p_i} &= 0 \\ \frac{\partial \mathcal{S}}{\partial x_i} &= p_i \\ \frac{\partial \mathcal{S}}{\partial t} &= -H \end{aligned}$$

Since the Hamiltonian is a given function of the phase space coordinates and time, $H = H(x_i, p_i, t)$, we combine the last two equations:

$$\frac{\partial \mathcal{S}}{\partial t} = -H(x_i, p_i, t) = -H(x_i, \frac{\partial \mathcal{S}}{\partial x_i}, t)$$

This first order differential equation in $s+1$ variables $(t, x_i; i = 1, \ldots s)$ for the principal function $\mathcal{S}$ is the Hamilton-Jacobi equation. Notice that the Hamilton-Jacobi equation has the same general form as the Schrödinger equation and is equally difficult to solve for all but special potentials. Nonetheless, we are guaranteed that a complete solution exists, and we will assume below that we can find it. Before proving our central theorem, we digress to examine the exact relationship between the Hamilton-Jacobi equation and the Schrödinger equation.

6.2 Quantum Mechanics and the Hamilton-Jacobi equation

The Hamiltonian-Jacobi equation provides the most direct link between classical and quantum mechanics. There is considerable similarity between the Hamilton-Jacobi equation and the Schrödinger equation:

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial t} &= -H(x_i, \frac{\partial \mathcal{S}}{\partial x_i}, t) \\ i\hbar \frac{\partial \psi}{\partial t} &= H(\hat{x}_i, \hat{p}_i, t) \end{aligned}$$

We make the relationship precise as follows.

Suppose the Hamiltonian in each case is that of a single particle in a potential:

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x})$$

Write the quantum wave function as

$$\psi = Ae^{\frac{i}{\hbar}\varphi}$$

The Schrödinger equation becomes

$$\begin{aligned} i\hbar \frac{\partial \left(Ae^{\frac{i}{\hbar}\varphi}\right)}{\partial t} &= -\frac{\hbar^2}{2m} \bigtriangledown^2 \left(Ae^{\frac{i}{\hbar}\varphi}\right) + V\left(Ae^{\frac{i}{\hbar}\varphi}\right) \\ i\hbar \frac{\partial A}{\partial t} e^{\frac{i}{\hbar}\varphi} - Ae^{\frac{i}{\hbar}\varphi} \frac{\partial \varphi}{\partial t} &= -\frac{\hbar^2}{2m} \bigtriangledown \cdot \left(e^{\frac{i}{\hbar}\varphi} \bigtriangledown A + \frac{i}{\hbar} Ae^{\frac{i}{\hbar}\varphi} \bigtriangledown \varphi\right) + VAe^{\frac{i}{\hbar}\varphi} \\ &= -\frac{\hbar^2}{2m} e^{\frac{i}{\hbar}\varphi} \left(\frac{i}{\hbar} \bigtriangledown \varphi \bigtriangledown A + \bigtriangledown^2 A\right) \\ &\quad -\frac{\hbar^2}{2m} e^{\frac{i}{\hbar}\varphi} \left(\frac{i}{\hbar} \bigtriangledown A \cdot \bigtriangledown\varphi + \frac{i}{\hbar} A \bigtriangledown^2 \varphi\right) \\ &\quad -\frac{\hbar^2}{2m} \left(\frac{i}{\hbar}\right)^2 e^{\frac{i}{\hbar}\varphi} \left(A \bigtriangledown \varphi \cdot \bigtriangledown\varphi\right) \\ &\quad +VAe^{\frac{i}{\hbar}\varphi} \end{aligned}$$

Then cancelling the exponential,

$$\begin{aligned} i\hbar\frac{\partial A}{\partial t} - A\frac{\partial \varphi}{\partial t} &= -\frac{i\hbar}{2m} \triangledown \varphi \triangledown A - \frac{\hbar^2}{2m} \triangledown^2 A \\ &\quad -\frac{i\hbar}{2m} \triangledown A \cdot \triangledown \varphi - \frac{i\hbar}{2m} A \triangledown^2 \varphi \\ &\quad +\frac{1}{2m} \left(A \triangledown \varphi \cdot \triangledown \varphi \right) + VA \end{aligned}$$

Collecting by powers of $\hbar$,

$$\begin{aligned} O\left(\hbar^0\right) &: \quad -\frac{\partial \varphi}{\partial t} = \frac{1}{2m} \triangledown \varphi \cdot \triangledown \varphi + V \\ O\left(\hbar^1\right) &: \quad \frac{1}{A}\frac{\partial A}{\partial t} = -\frac{1}{2m}\left(\frac{2}{A} \triangledown A \cdot \triangledown \varphi + \triangledown^2 \varphi \right) \\ O\left(\hbar^2\right) &: \quad 0 = -\frac{\hbar^2}{2m} \triangledown^2 A \end{aligned}$$

The zeroth order terms is the Hamilton-Jacobi equation, with $\varphi = \mathcal{S}$:

$$\begin{aligned} -\frac{\partial \mathcal{S}}{\partial t} &= \frac{1}{2m} \triangledown \mathcal{S} \cdot \triangledown \mathcal{S} + V \\ &= \frac{1}{2m}\mathbf{p}^2 + V(x) \end{aligned}$$

where $p = \triangledown \mathcal{S}$. Therefore, the Hamilton-Jacobi equation is the $\hbar \rightarrow 0$ limit of the Schrödinger equation.

6.3 Trivialization of the motion

We now seek a solution, in principle, to the complete mechanical problem. The solution is to find a canonical transformation that makes the motion trivial. Hamilton's principal function, the solution to the Hamilton-Jacobi equation, is the generating function of this canonical transformation.

To begin, suppose we have a solution to the Hamilton-Jacobi equation of the form

$$\mathcal{S} = g(t, x_1, \ldots, x_s, \alpha_1, \ldots, \alpha_s) + A$$

where the α_i and A provide $s+1$ constants describing the solution. Such a solution is called a *complete* integral of the equation, as opposed to a *general* integral which depends on arbitrary functions. We will show below that a complete solution leads to a general solution. We use $\mathcal{S}$ as a generating function.

Our canonical transformation will take the variables (x_i, p_i) to a new set of variables $\left(\beta^i, \alpha_i\right)$. Since $\mathcal{S}$ depends on the old coordinates x_i and the new momenta α_i, we have the relations

$$\begin{aligned} p_i &= \frac{\partial \mathcal{S}}{\partial x_i} \\ \beta_i &= \frac{\partial \mathcal{S}}{\partial \alpha_i} \\ H' &= H + \frac{\partial \mathcal{S}}{\partial t} \end{aligned}$$

Notice that the new Hamiltonian, H', vanishes because the Hamiltonian-Jacobi equation is satisfied by $\mathcal{S}$!. With $H' = 0$, Hamilton's equations in the new canonical coordinates are simply

$$\begin{aligned} \frac{d\alpha_i}{dt} &= \frac{\partial H'}{\partial \beta_i} = 0 \\ \frac{d\beta_i}{dt} &= -\frac{\partial H'}{\partial \alpha_i} = 0 \end{aligned}$$

with solutions

$$\begin{aligned}\alpha_i &= const.\\ \beta_i &= const.\end{aligned}$$

The system remains at the phase space point (α_i, β_i). To find the motion in the original coordinates as functions of time and the $2s$ constants of motion,

$$x_i = x_i(t; \alpha_i, \beta_i)$$

we can algebraically invert the s equations

$$\beta_i = \frac{\partial g(x_i, t, \alpha_i)}{\partial \alpha_i}$$

The momenta may be found by differentiating the principal function,

$$p_i = \frac{\partial \mathcal{S}(x_i, t, \alpha_i)}{\partial x_i}$$

Therefore, solving the Hamilton-Jacobi equation is the key to solving the full mechanical problem. Furthermore, we know that a solution exists because Hamilton's equations satisfy the integrability equation for $\mathcal{S}$.

We note one further result. While we have made use of a *complete* integral to solve the mechanical problem, we may want a *general* integral of the Hamilton-Jacobi equation. The difference is that a complete integral of an equation in $s+1$ variables depends on $s+1$ constants, while a general integral depends on s *functions*. Fortunately, a complete integral of the equation can be used to construct a general integral, and there is no loss of generality in considering a complete integral. We see this as follows. A complete solution takes the form

$$\mathcal{S} = g(t, x_1, \ldots, x_s, \alpha_1, \ldots, \alpha_s) + A$$

To find a general solution, think of the constant A as a function of the other s constants, $A(\alpha_1, \ldots, \alpha_s)$. Now replace each of the α_i by a function of the coordinates and time, $\alpha_i \to h_i(t, x_i)$. This makes $\mathcal{S}$ depend on arbitrary functions, but we need to make sure it still solves the Hamilton-Jacobi equation. It will provided the partials of $\mathcal{S}$ with respect to the coordinates remain unchanged. In general, these partials are given by

$$\frac{\partial \mathcal{S}}{\partial x_i} = \left(\frac{\partial \mathcal{S}}{\partial x_i}\right)_{h_i = const.} + \left(\frac{\partial \mathcal{S}}{\partial h_k}\right)_{x = const.} \frac{\partial h_k}{\partial x_i}$$

We therefore still have solutions provided

$$\left(\frac{\partial \mathcal{S}}{\partial h_k}\right)_{x = const.} \frac{\partial h_k}{\partial x_i} = 0$$

and since we want h_k to be an arbitrary function of the coordinates, we demand

$$\left(\frac{\partial \mathcal{S}}{\partial h_k}\right)_{x = const.} = 0$$

Then

$$\frac{\partial \mathcal{S}}{\partial h_k} = \frac{\partial}{\partial h_k}\left(g(t, x_i, \alpha_i) + A(\alpha_i)\right) = 0$$

and we have

$$A(\alpha_1, \ldots, \alpha_s) = const. - g$$

This just makes A into some specific function of x^i and t.

Since the partials with respect to the coordinates are the same, and we haven't changed the time dependence,

$$\mathcal{S} = g(t, x_1, \ldots, x_s, h_1, \ldots, h_s) + A\left(h_i\right)$$

is a general solution to the Hamilton-Jacobi equation.

6.3.1 Example 1: Free particle

The simplest example is the case of a free particle, for which the Hamiltonian is

$$H = \frac{p^2}{2m}$$

and the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}\left(S'\right)^2$$

Let

$$S = f(x) - Et$$

Then $f(x)$ must satisfy

$$\frac{df}{dx} = \sqrt{2mE}$$

and therefore

$$\begin{aligned} f(x) &= \sqrt{2mE}x - c \\ &= \pi x - c \end{aligned}$$

where c is constant and we write the integration constant E in terms of the new (constant) momentum. Hamilton's principal function is therefore

$$S\left(x, \pi, t\right) = \pi x - \frac{\pi^2}{2m}t - c$$

Then, for a generating function of this type we have

$$\begin{aligned} p &= \frac{\partial S}{\partial x} = \pi \\ q &= \frac{\partial S}{\partial \pi} = x - \frac{\pi}{m}t \\ H' &= H + \frac{\partial S}{\partial t} = H - E \end{aligned}$$

Because $E = H$, the new Hamiltonian, H', is zero. This means that both q and π are constant. The solution for x and p follows immediately:

$$\begin{aligned} x &= q + \frac{\pi}{m}t \\ p &= \pi \end{aligned}$$

We see that the new canonical variables (q, π) are just the initial position and momentum of the motion, and therefore do determine the motion. The fact that knowing q and π is equivalent to knowing the full motion rests here on the fact that S generates motion along the classical path. In fact, given initial conditions (q, π), we can use Hamilton's principal function as a generating function but treat π as the *old* momentum and x as the *new* coordinate to reverse the process above and generate $x(t)$ and p.

6.3.2 Example 2: Simple harmonic oscillator

For the simple harmonic oscillator, the Hamiltonian becomes

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

and the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}\left(S'\right)^2 + \frac{1}{2}kx^2$$

Letting

$$S = f(x) - Et$$

as before, $f(x)$ must satisfy

$$\frac{df}{dx} = \sqrt{2m\left(E - \frac{1}{2}kx^2\right)}$$

and therefore

$$\begin{aligned} f(x) &= \int \sqrt{2m\left(E - \frac{1}{2}kx^2\right)}dx \\ &= \int \sqrt{\pi^2 - mkx^2}dx \end{aligned}$$

where we have set $E = \frac{\pi^2}{2m}$. Now let $\sqrt{mk}x = \pi \sin y$. The integral is immediate:

$$\begin{aligned} f(x) &= \int \sqrt{\pi^2 - mkx^2}dx \\ &= \frac{\pi^2}{\sqrt{mk}} \int \cos^2 y dy \\ &= \frac{\pi^2}{2\sqrt{mk}}\left(y + \sin y \cos y\right) \end{aligned}$$

Hamilton's principal function is therefore

$$\begin{aligned} S(x, \pi, t) &= \frac{\pi^2}{2\sqrt{mk}}\left(\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) + \sqrt{mk}\frac{x}{\pi}\sqrt{1 - mk\frac{x^2}{\pi^2}}\right) \\ &\quad -\frac{\pi^2}{2m}t - c \\ &= \frac{\pi^2}{2\sqrt{mk}}\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) + \frac{x}{2}\sqrt{\pi^2 - mkx^2} - \frac{\pi^2}{2m}t - c \end{aligned}$$

and we may use it to generate the canonical change of variable.

This time we have

$$\begin{aligned} p &= \frac{\partial S}{\partial x} \\ &= \frac{\pi}{2}\frac{1}{\sqrt{1 - mk\frac{x^2}{\pi^2}}} + \frac{1}{2}\sqrt{\pi^2 - mkx^2} + \frac{x}{2}\frac{-mkx}{\sqrt{\pi^2 - mkx^2}} \\ &= \frac{1}{\sqrt{\pi^2 - mkx^2}}\left(\frac{\pi^2}{2} + \frac{1}{2}\left(\pi^2 - mkx^2\right) - \frac{mkx^2}{2}\right) \\ &= \sqrt{\pi^2 - mkx^2} \\ q &= \frac{\partial S}{\partial \pi} \\ &= \frac{\pi}{\sqrt{mk}}\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) + \frac{\pi^2}{2\sqrt{mk}}\frac{1}{\sqrt{1 - mk\frac{x^2}{\pi^2}}}\left(-\sqrt{mk}\frac{x}{\pi^2}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{x}{2}\frac{\pi}{\sqrt{\pi^2 - mkx^2}} - \frac{\pi}{m}t \\
&= \frac{\pi}{\sqrt{mk}}\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) - \frac{\pi}{m}t \\
H' &= H + \frac{\partial S}{\partial t} = H - E = 0
\end{aligned}
$$

The first equation relates p to the energy and position, the second gives the new position coordinate q, and third equation shows that the new Hamiltonian is zero. Hamilton's equations are trivial, so that π and q are constant, and we can invert the expression for q to give the solution. Setting $\omega = \sqrt{\frac{k}{m}}$, the solution is

$$
\begin{aligned}
x(t) &= \frac{\pi}{m\omega}\sin\left(\frac{m\omega}{\pi}q + \omega t\right) \\
&= A\sin(\omega t + \phi)
\end{aligned}
$$

where

$$
\begin{aligned}
q &= A\phi \\
\pi &= Am\omega
\end{aligned}
$$

The new canonical coordinates therefore characterize the initial amplitude and phase of the oscillator.

6.3.3 Example 3: One dimensional particle motion

Now suppose a particle with one degree of freedom moves in a potential $U(x)$. Little is changed. The the Hamiltonian becomes

$$H = \frac{p^2}{2m} + U$$

and the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}(S')^2 + U(x)$$

Letting

$$S = f(x) - Et$$

as before, $f(x)$ must satisfy

$$\frac{df}{dx} = \sqrt{2m(E - U(x))}$$

and therefore

$$
\begin{aligned}
f(x) &= \int \sqrt{2m(E - U(x))}dx \\
&= \int \sqrt{\pi^2 - 2mU(x)}dx
\end{aligned}
$$

where we have set $E = \frac{\pi^2}{2m}$. Hamilton's principal function is therefore

$$S(x, \pi, t) = \int \sqrt{\pi^2 - 2mU(x)}dx - \frac{\pi^2}{2m}t - c$$

and we may use it to generate the canonical change of variable.

This time we have

$$\begin{aligned}
p &= \frac{\partial S}{\partial x} = \sqrt{\pi^2 - 2mU(x)} \\
q &= \frac{\partial S}{\partial \pi} = \frac{\partial}{\partial \pi}\left(\int_{x_0}^{x} \sqrt{\pi^2 - 2mU(x)}dx\right) - \frac{\pi}{m}t \\
H' &= H + \frac{\partial S}{\partial t} = H - E = 0
\end{aligned}$$

The first and third equations are as expected, while for q we may interchange the order of differentiation and integration:

$$\begin{aligned}
q &= \frac{\partial}{\partial \pi}\left(\int \sqrt{\pi^2 - 2mU(x)}dx\right) - \frac{\pi}{m}t \\
&= \int \frac{\partial}{\partial \pi}\left(\sqrt{\pi^2 - 2mU(x)}\right) dx - \frac{\pi}{m}t \\
&= \int \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}t
\end{aligned}$$

To complete the problem, we need to know the potential. However, even without knowing $U(x)$ we can make sense of this result by combining the expression for q above to our previous solution to the same problem. There, conservation of energy gives a first integral to Newton's second law,

$$\begin{aligned}
E &= \frac{p^2}{2m} + U \\
&= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U
\end{aligned}$$

so we arrive at the familiar quadrature

$$t - t_0 = \int dt = \int_{x_0}^{x} \frac{mdx}{\sqrt{2m(E-U)}}$$

Substituting into the expression for q,

$$\begin{aligned}
q &= \int^{x} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}\int_{x_0}^{x} \frac{mdx}{\sqrt{2m(E-U)}} - \frac{\pi}{m}t_0 \\
&= \int^{x} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \int_{x_0}^{x} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}t_0 \\
&= \int^{x_0} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}t_0
\end{aligned}$$

We once again find that q is a constant characterizing the initial configuration. Since t_0 is the time at which the position is x_0 and the momentum is p_0, we have the following relations:

$$\frac{p^2}{2m} + U(x) = \frac{p_0^2}{2m} + U(x_0) = E = const.$$

and

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E-U)}}$$

which we may rewrite as

$$t - \int^{x} \frac{dx}{\sqrt{\frac{2}{m}(E-U)}} = t_0 - \int^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E-U)}} = \frac{m}{\pi}q = const.$$