Hamilton-Jacobi theory

November 29, 2014

We conclude with the crowning theorem of Hamiltonian dynamics: a proof that for any Hamiltonian dynamical system there exists a canonical transformation to a set of variables on phase space such that the paths of motion reduce to single points. Clearly, this theorem shows the power of canonical transformations! The theorem relies on describing solutions to the Hamilton-Jacobi equation, which we introduce first.

1 Integrability of the action

We first define Hamilton's principal function. Let $x^i(t)$ and $p_i(t)$ satisfy Hamilton's equations of motion, and ask for the integrability condition for the action. That is, we would like to know when the action is a function and not a functional, $S\left[x^i(t)\right] \Rightarrow S\left(x^i, t\right)$. The condition we need is just like the vanishing curl of a force required for the existence of a potential function. Thinking of the $n+1$ vector$P_a = (p_i, -H)$ integrated along a curve in $dX^a = \left(x^i, t\right)$-space

$$S = \int p_i dx^i - H dt = \int P_a dX^a$$

the integrability condition is the vanishing of the higher-dimensional curl,

$$\frac{\partial P_a}{\partial X^b} - \frac{\partial P_b}{\partial X^a} = 0$$

Writing this in terms of p_i, H, x^j, t,

$$\begin{aligned}
\frac{\partial p_i}{\partial x^j} - \frac{\partial p_j}{\partial x^i} &= 0 \\
-\frac{\partial H}{\partial x^i} - \frac{dp_i}{dt} &= 0 \\
\frac{dp_i}{dt} + \frac{\partial H}{\partial x^i} &= 0 \\
-\frac{\partial H}{\partial t} + \frac{\partial H}{\partial t} &= 0
\end{aligned}$$

The first is satisfied because x^i and p_j are independent, the middle two give one of Hamilton's equations, and the final equation is an identity. Therefore, S is a function if $\dot{p}_i = -\frac{\partial H}{\partial x^i}$.

The condition is not unique. Since we may integrate by parts,

$$\begin{aligned}
S &= \int p_i dx^i - H dt \\
&= \int d\left(p_i x^i\right) - x^i dp_i - H dt \\
&= p_i x^i \Big|_{t_0}^{t_1} - \int \left(x^i dp_i + H dt\right)
\end{aligned}$$

a similar argument applied to $\int \left(x^i dp_i + H dt \right)$ shows that $\dot{x}^i = \frac{\partial H}{\partial p_i}$ gives integrability. Therefore, if the family of curves, $\left(x^i (t), p_j (t) \right)$ solve Hamilton's equations, then evaluating the action on those curves gives a function.

Exercise: Carry out the demonstration that S becomes a function if $\dot{x}^i = \frac{\partial H}{\partial p_i}$

2 The Hamilton-Jacobi equation

Conversely, suppose we replace the action with a function, $S[x] \rightarrow \mathcal{S}\left(x^i, t\right)$. Then

$$\mathcal{S}\left(x^i, t\right) = \int p_i dx^i - H dt$$

implies

$$\begin{aligned} d\mathcal{S} &= p_i dx^i - H dt \\ \frac{\partial \mathcal{S}}{\partial x^i} dx^i + \frac{\partial \mathcal{S}}{\partial t} dt &= p_i dx^i - H dt \end{aligned}$$

so that

$$\begin{aligned} p_i &= \frac{\partial \mathcal{S}}{\partial x^i} \\ H\left(x^i, p_j, t\right) &= -\frac{\partial \mathcal{S}}{\partial t} \end{aligned}$$

If we replace p_j in the Hamiltonian, we get a differential equation for Hamilton's principal function,

$$H\left(x^i, \frac{\partial \mathcal{S}}{\partial x^j}, t\right) = -\frac{\partial \mathcal{S}}{\partial t}$$

This is the *Hamilton-Jacobi* equation.

Example: Find the Hamilton-Jacobi equation for a simple harmonic oscillator Since the Hamiltonian for the oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right) + \frac{1}{2} k x^2 = -\frac{\partial \mathcal{S}}{\partial t}$$

Partial differential equations have free functions in their solutions. Thus, while

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 0$$

has the solution

$$f(x, t) = ax + by + c$$

it has the general solution

$$f(x, y) = g_-(x + t) + g_+(x - t)$$

for *any* two functions $g_\pm$.

For the Hamilton-Jacobi equation, canonical transformations can help introduce arbitrary functions. Suppose we have a solution for $\mathcal{S}$ of the form

$$\mathcal{S} = g\left(x^i, \alpha_j, t\right)$$

for some function g. Then a canonical transformation changes $p_i dx^i - H dt$ by df, where f is the generating function. This means that $\mathcal{S}\left(X^i, P_j, t\right) = g\left(x^i, \alpha_j, t\right) + f$, where the new coordinates are X^i, P_j. Now let

$$f = f\left(X^i, \alpha_j\right) - x^i \alpha_i$$

Then we have

$$\begin{aligned}
\alpha_i dx^i - H dt &= P_i dX^i - H' dt + df \\
&= P_i dX^i - H' dt - x^i d\alpha_i - \alpha_i dx^i + \frac{\partial f}{\partial X^i} dX^i + \frac{\partial f}{\partial \alpha_i} d\alpha_i + \frac{\partial f}{\partial t} dt \\
&= \left(P_i + \frac{\partial f}{\partial X^i}\right) dX^i + \left(\frac{\partial f}{\partial \alpha_i} - x^i\right) d\alpha_i + \left(H - H'\right) dt
\end{aligned}$$

so that

$$\begin{aligned}
P_i &= -\frac{\partial f\left(X, \alpha\right)}{\partial X^i} \\
x^i &= \frac{\partial f\left(X, \alpha\right)}{\partial \alpha_i} \\
H' &= H
\end{aligned}$$

The Hamiltonian remains unchanged, the new coordinate X^i is found by inverting $x^i = \frac{\partial f(X,\alpha)}{\partial \alpha_i}$ to find x^i as a function of the new coordinates and the constants α_i. The new momentum P_i is an arbitrary function of the new coordinates X^i, and the principal function becomes

$$\begin{aligned}
\mathcal{S}\left(X^i, t\right) &= g\left(x^i, \alpha_j, t\right) + f \\
&= g\left(x^i\left(X, \alpha\left(X, P\right)\right), \alpha\left(X, P\right), t\right) + f\left(X^i, \alpha_j\left(X, P\right)\right) - x^i\left(X, P\right) \alpha_i\left(X, P\right)
\end{aligned}$$

so the new form of the principal function depends on n arbitrary functions $P_i\left(x, \alpha\right)$.

3 The principal function as generator of a canonical transformation

Suppose we find Hamilton's principal function, $\mathcal{S}\left(x^i, t\right)$. Its relationship to the momentum, $p_i = \frac{\partial \mathcal{S}}{\partial x^i}$, suggests that we may use it as a generating function for a canonical transformation, $\mathcal{S}\left(x^i, P_j, t\right) = \mathcal{S}\left(x^i, t\right)$. This turns out to be especially useful.

We choose a generating function with independent variables x^i, P_j, so let

$$f = -X^i P_i + \mathcal{S}\left(x^i, t\right)$$

Then we have

$$\begin{aligned}
p_i dx^i - H dt &= P_i dX^i - H' dt + df \\
&= P_i dX^i - H' dt - X^i dP_i - P_i dX^i + \frac{\partial \mathcal{S}}{\partial x^i} dx^i + \frac{\partial \mathcal{S}}{\partial P_i} dP_i + \frac{\partial \mathcal{S}}{\partial t} dt \\
&= \left(\frac{\partial \mathcal{S}}{\partial x^i} - p_i\right) dx^i + \left(\frac{\partial \mathcal{S}}{\partial P_i} - X^i\right) dP_i + \left(H + \frac{\partial \mathcal{S}}{\partial t} - H'\right) dt
\end{aligned}$$

so that the independent variables are now $\left(x^i, P_i, t\right)$, satisfying

$$\begin{aligned}
X^i &= \frac{\partial \mathcal{S}}{\partial P_i} = 0 \\
p_i &= \frac{\partial \mathcal{S}}{\partial x^i} \\
H' &= H + \frac{\partial \mathcal{S}}{\partial t}
\end{aligned}$$

But we know that

$$\begin{aligned}
\frac{\partial \mathcal{S}}{\partial x^i} &= p_i \\
\frac{\partial \mathcal{S}}{\partial p_i} &= 0 \\
\frac{\partial \mathcal{S}}{\partial t} &= -H
\end{aligned}$$

so that $H' = 0$.

The principal function has generated a transformation to a set of canonical variables for which the Hamiltonian vanishes! This makes Hamilton's equations trivial:

$$\begin{aligned}
\dot{X}^i &= 0 \\
\dot{P}_i &= 0
\end{aligned}$$

so $\left(X^i, P_j\right)$ simply stay at their initial values.

4 Examples

4.1 Example 1: Free particle

The simplest example is the case of a free particle, for which the Hamiltonian is

$$H = \frac{p^2}{2m}$$

and the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}\left(S'\right)^2$$

Let

$$S = f(x) - Et$$

Then $f(x)$ must satisfy

$$\frac{df}{dx} = \sqrt{2mE}$$

and therefore

$$\begin{aligned}
f(x) &= \sqrt{2mE}x - c \\
&= \pi x - c
\end{aligned}$$

where c is constant and we write the integration constant E in terms of the new (constant) momentum. Hamilton's principal function is therefore

$$S(x, \pi, t) = \pi x - \frac{\pi^2}{2m}t - c$$

Then, for a generating function of this type we have

$$\begin{aligned}
p &= \frac{\partial S}{\partial x} = \pi \\
q &= \frac{\partial S}{\partial \pi} = x - \frac{\pi}{m}t \\
H' &= H + \frac{\partial S}{\partial t} = H - E
\end{aligned}$$

Because $E = H$, the new Hamiltonian, H', is zero. This means that both q and π are constant. The solution for x and p follows immediately:

$$\begin{aligned} x &= q + \frac{\pi}{m} t \\ p &= \pi \end{aligned}$$

We see that the new canonical variables (q, π) are just the initial position and momentum of the motion, and therefore do determine the motion. The fact that knowing q and π is equivalent to knowing the full motion rests here on the fact that S generates motion along the classical path. In fact, given initial conditions (q, π), we can use Hamilton's principal function as a generating function but treat π as the *old* momentum and x as the *new* coordinate to reverse the process above and generate $x(t)$ and p.

4.2 Example 2: Simple harmonic oscillator

For the simple harmonic oscillator, the Hamiltonian becomes

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

and the Hamilton-Jacobi equation is

$$\frac{1}{2m} \left(\frac{\partial \mathcal{S}}{\partial x} \right)^2 + \frac{1}{2} k x^2 = -\frac{\partial \mathcal{S}}{\partial t}$$

Setting $\mathcal{S} = f(x) - Et$ then

$$\begin{aligned} \frac{1}{2m} \left(\frac{df}{dx} \right)^2 + \frac{1}{2} k x^2 &= E \\ \frac{df}{dx} &= \sqrt{2mE - mkx^2} \end{aligned}$$

and direct integration (see examples, below) give a solution for $\mathcal{S}$,

$$\begin{aligned} f &= \int \sqrt{2mE - mkx^2} dx \\ &= \sqrt{2mE} \int \sqrt{1 - \frac{k}{2E} x^2} dx \end{aligned}$$

so with $\sqrt{\frac{k}{2E}} x = \sin \mu$ we have

$$\begin{aligned} f &= \sqrt{2mE} \int \sqrt{1 - \frac{k}{2E} \sin^2 \mu} \sqrt{\frac{2E}{k}} \cos \mu d\mu \\ &= \frac{2E}{\omega} \int \cos^2 \mu d\mu \\ &= \frac{2E}{\omega} \int \frac{1}{2} (1 + \cos 2\mu) d\mu \\ &= \frac{E}{\omega} \left(\mu + \frac{1}{2} \sin 2\mu \right) \\ &= \frac{E}{\omega} \left(\sin^{-1} \sqrt{\frac{k}{2E}} x + \sin \left(\sin^{-1} \sqrt{\frac{k}{2E}} x \right) \cos \left(\sin^{-1} \sqrt{\frac{k}{2E}} x \right) \right) \\ &= \frac{E}{\omega} \left(\sin^{-1} \sqrt{\frac{k}{2E}} x + \sqrt{\frac{k}{2E}} x \sqrt{1 - \frac{k}{2E} x^2} \right) \end{aligned}$$

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}\left(S'\right)^2 + \frac{1}{2}kx^2$$

Letting

$$S = f(x) - Et$$

as before, $f(x)$ must satisfy

$$\frac{df}{dx} = \sqrt{2m\left(E - \frac{1}{2}kx^2\right)}$$

and therefore

$$\begin{aligned} f(x) &= \int \sqrt{2m\left(E - \frac{1}{2}kx^2\right)}dx \\ &= \int \sqrt{\pi^2 - mkx^2}dx \end{aligned}$$

where we have set $E = \frac{\pi^2}{2m}$. Now let $\sqrt{mk}x = \pi \sin y$. The integral is immediate:

$$\begin{aligned} f(x) &= \int \sqrt{\pi^2 - mkx^2}dx \\ &= \frac{\pi^2}{\sqrt{mk}}\int \cos^2 y dy \\ &= \frac{\pi^2}{2\sqrt{mk}}\left(y + \sin y \cos y\right) \end{aligned}$$

Hamilton's principal function is therefore

$$\begin{aligned} S\left(x,\pi,t\right) &= \frac{\pi^2}{2\sqrt{mk}}\left(\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) + \sqrt{mk}\frac{x}{\pi}\sqrt{1 - mk\frac{x^2}{\pi^2}}\right) \\ &\quad -\frac{\pi^2}{2m}t - c \\ &= \frac{\pi^2}{2\sqrt{mk}}\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) + \frac{x}{2}\sqrt{\pi^2 - mkx^2} - \frac{\pi^2}{2m}t - c \end{aligned}$$

and we may use it to generate the canonical change of variable.

This time we have

$$\begin{aligned} p &= \frac{\partial S}{\partial x} \\ &= \frac{\pi}{2}\frac{1}{\sqrt{1 - mk\frac{x^2}{\pi^2}}} + \frac{1}{2}\sqrt{\pi^2 - mkx^2} + \frac{x}{2}\frac{-mkx}{\sqrt{\pi^2 - mkx^2}} \\ &= \frac{1}{\sqrt{\pi^2 - mkx^2}}\left(\frac{\pi^2}{2} + \frac{1}{2}\left(\pi^2 - mkx^2\right) - \frac{mkx^2}{2}\right) \\ &= \sqrt{\pi^2 - mkx^2} \\ q &= \frac{\partial S}{\partial \pi} \\ &= \frac{\pi}{\sqrt{mk}}\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) + \frac{\pi^2}{2\sqrt{mk}}\frac{1}{\sqrt{1 - mk\frac{x^2}{\pi^2}}}\left(-\sqrt{mk}\frac{x}{\pi^2}\right) \end{aligned}$$

$$
\begin{aligned}
& +\frac{x}{2}\frac{\pi}{\sqrt{\pi^2 - mkx^2}} - \frac{\pi}{m}t \\
= \; & \frac{\pi}{\sqrt{mk}}\sin^{-1}\left(\sqrt{mk}\frac{x}{\pi}\right) - \frac{\pi}{m}t \\
H' = \; & H + \frac{\partial S}{\partial t} = H - E = 0
\end{aligned}
$$

The first equation relates p to the energy and position, the second gives the new position coordinate q, and third equation shows that the new Hamiltonian is zero. Hamilton's equations are trivial, so that π and q are constant, and we can invert the expression for q to give the solution. Setting $\omega = \sqrt{\frac{k}{m}}$, the solution is

$$
\begin{aligned}
x(t) &= \frac{\pi}{m\omega}\sin\left(\frac{m\omega}{\pi}q + \omega t\right) \\
&= A\sin(\omega t + \phi)
\end{aligned}
$$

where

$$
\begin{aligned}
q &= A\phi \\
\pi &= Am\omega
\end{aligned}
$$

The new canonical coordinates therefore characterize the initial amplitude and phase of the oscillator.

4.3 Example 3: One dimensional particle motion

Now suppose a particle with one degree of freedom moves in a potential $U(x)$. Little is changed. The the Hamiltonian becomes

$$H = \frac{p^2}{2m} + U$$

and the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}(S')^2 + U(x)$$

Letting

$$S = f(x) - Et$$

as before, $f(x)$ must satisfy

$$\frac{df}{dx} = \sqrt{2m(E - U(x))}$$

and therefore

$$
\begin{aligned}
f(x) &= \int \sqrt{2m(E - U(x))}dx \\
&= \int \sqrt{\pi^2 - 2mU(x)}dx
\end{aligned}
$$

where we have set $E = \frac{\pi^2}{2m}$. Hamilton's principal function is therefore

$$S(x, \pi, t) = \int \sqrt{\pi^2 - 2mU(x)}dx - \frac{\pi^2}{2m}t - c$$

and we may use it to generate the canonical change of variable.

This time we have

$$
\begin{aligned}
p &= \frac{\partial S}{\partial x} = \sqrt{\pi^2 - 2mU(x)} \\
q &= \frac{\partial S}{\partial \pi} = \frac{\partial}{\partial \pi}\left(\int_{x_0}^{x} \sqrt{\pi^2 - 2mU(x)}dx\right) - \frac{\pi}{m}t \\
H' &= H + \frac{\partial S}{\partial t} = H - E = 0
\end{aligned}
$$

The first and third equations are as expected, while for q we may interchange the order of differentiation and integration:

$$
\begin{aligned}
q &= \frac{\partial}{\partial \pi}\left(\int \sqrt{\pi^2 - 2mU(x)}dx\right) - \frac{\pi}{m}t \\
&= \int \frac{\partial}{\partial \pi}\left(\sqrt{\pi^2 - 2mU(x)}\right) dx - \frac{\pi}{m}t \\
&= \int \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}t
\end{aligned}
$$

To complete the problem, we need to know the potential. However, even without knowing $U(x)$ we can make sense of this result by combining the expression for q above to our previous solution to the same problem. There, conservation of energy gives a first integral to Newton's second law,

$$
\begin{aligned}
E &= \frac{p^2}{2m} + U \\
&= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U
\end{aligned}
$$

so we arrive at the familiar quadrature

$$
t - t_0 = \int dt = \int_{x_0}^{x} \frac{mdx}{\sqrt{2m(E-U)}}
$$

Substituting into the expression for q,

$$
\begin{aligned}
q &= \int^{x} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}\int_{x_0}^{x} \frac{mdx}{\sqrt{2m(E-U)}} - \frac{\pi}{m}t_0 \\
&= \int^{x} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \int_{x_0}^{x} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}t_0 \\
&= \int^{x_0} \frac{\pi dx}{\sqrt{\pi^2 - 2mU(x)}} - \frac{\pi}{m}t_0
\end{aligned}
$$

We once again find that q is a constant characterizing the initial configuration. Since t_0 is the time at which the position is x_0 and the momentum is p_0, we have the following relations:

$$
\frac{p^2}{2m} + U(x) = \frac{p_0^2}{2m} + U(x_0) = E = const.
$$

and

$$
t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E-U)}}
$$

which we may rewrite as

$$
t - \int^{x} \frac{dx}{\sqrt{\frac{2}{m}(E-U)}} = t_0 - \int^{x_0} \frac{dx}{\sqrt{\frac{2}{m}(E-U)}} = \frac{m}{\pi}q = const.
$$

4.4 Example 4: Two dimensional oscillator

Suppose we have a mass fastened to a spring, moving on a tabletop, so that action is

$$S = \int \left(\frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) - \frac{1}{2} k r^2 \right)$$

The canonical momenta are

$$\begin{aligned} p_r &= m\dot{r} \\ p_\varphi &= mr^2 \dot{\varphi} \end{aligned}$$

so the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) + \frac{1}{2} k r^2 \\ H &= \frac{1}{2m} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) + \frac{1}{2} k r^2 \end{aligned}$$

The Hamilton-Jacobi equation is

$$\frac{1}{2m} \left(\left(\frac{\partial \mathcal{S}}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \mathcal{S}}{\partial \varphi} \right)^2 \right) + \frac{1}{2} k r^2 = -\frac{\partial \mathcal{S}}{\partial t}$$

Let $\mathcal{S} = f(r, \varphi) - Et$ so that

$$\frac{1}{2m} \left(\left(\frac{\partial \mathcal{S}}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \mathcal{S}}{\partial \varphi} \right)^2 \right) + \frac{1}{2} k r^2 = E$$

Now use separation of variables. Let

$$f = R(r) + \Phi(\varphi)$$

Then

$$\begin{aligned} \left(\frac{dR}{dr} \right)^2 + \frac{1}{r^2} \left(\frac{d\Phi}{d\varphi} \right)^2 + kmr^2 &= 2mE \\ \left(\frac{dR}{dr} \right)^2 + kmr^2 - 2mE &= -\frac{1}{r^2} \left(\frac{d\Phi}{d\varphi} \right)^2 \\ -r^2 \left(\frac{dR}{dr} \right)^2 - kmr^4 + 2mEr^2 &= \left(\frac{d\Phi}{d\varphi} \right)^2 \end{aligned}$$

Since the left side depends only on r and the right only on φ, each side must equal some constant, a^2:

$$\begin{aligned} -r^2 \left(\frac{dR}{dr} \right)^2 - kmr^4 + 2mEr^2 - a &= 0 \\ \frac{d\Phi}{d\varphi} &= \pm a \end{aligned}$$

We immediately have

$$\Phi = \pm a\varphi + b$$

and we may integrate to find R:

$$\begin{aligned} \frac{dR}{dr} &= \sqrt{-kmr^2 + 2mE - \frac{a}{r^2}} \\ R &= \int \sqrt{2mE - kmr^2 - \frac{a}{r^2}} dr \end{aligned}$$