Central Forces II: Gravitation

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1 Solving the equation of motion

We have shown that the action for any two body system acted on by a central force may be written as

$$S = \int_0^t \left(\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r) \right) dt$$

where $\mu = \frac{mM}{M+m}$ is the reduced mass and $L = \mu r^2\dot{\varphi}$ the conserved angular momentum.

The equation of motion was found to be

$$\mu\ddot{r} - \frac{L^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0$$

but we work instead with the conserved energy,

$$\begin{aligned} E &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) \end{aligned}$$

Notice that we may have $E < 0$. The energy is fixed by its initial value. Taking $r = r_{min}$ for a bounded orbit at $t = 0$,

$$E = \frac{L^2}{2\mu r_{min}^2} + V(r_{min})$$

1.1 Additional conserved quantitites

From the angular momentum and the energy we may construct another conserved quantity. The time rate of change of the unit vector $\hat{\boldsymbol{\varphi}}$ is given by

$$\begin{aligned} \frac{d}{dt}\hat{\boldsymbol{\varphi}} &= \frac{d}{dt}\left(-\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi\right) \\ &= \left(-\mathbf{i}\cos\varphi - \mathbf{j}\sin\varphi\right)\dot{\varphi} \\ &= -\dot{\varphi}\hat{\mathbf{r}} \end{aligned}$$

and therefore, using $L = \mu r^2\dot{\varphi}$, we have

$$\begin{aligned} \frac{d}{dt}\hat{\boldsymbol{\varphi}} &= -\frac{L}{\mu r^2}\hat{\mathbf{r}} \\ &= \frac{L}{\mu\alpha}\mathbf{F} \\ &= \frac{d}{dt}\left(\frac{L}{\mu\alpha}\mathbf{p}\right) \end{aligned}$$

where the force is given by $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \equiv -\frac{\alpha}{r^2}\hat{\mathbf{r}}$and we have

$$\frac{d}{dt}\left(\mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}\right) = 0$$

Therefore, *Hamilton's vector*,

$$\mathbf{h} = \mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}$$

is conserved as a consequence of rotational invariance.

Since angular momentum is conserved, the product

$$\begin{aligned}
\mathbf{A} &\equiv \mathbf{h} \times \mathbf{L} \\
&= \left(\mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}\right) \times \mathbf{L} \\
&= \mathbf{p} \times \mathbf{L} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}} \times (\mathbf{r} \times \mathbf{p}) \\
&= \mathbf{p} \times \mathbf{L} - \frac{\mu\alpha}{L}\left(\mathbf{r}\left(\hat{\boldsymbol{\varphi}} \cdot \mathbf{p}\right) - \mathbf{p}\left(\hat{\boldsymbol{\varphi}} \cdot \mathbf{r}\right)\right) \\
&= \mathbf{p} \times \mathbf{L} - \frac{\mu\alpha}{L}\left(\mu r^2 \dot{\varphi}\right)\mathbf{r} \\
&= \mathbf{p} \times \mathbf{L} - \mu\alpha\mathbf{r}
\end{aligned}$$

as also conserved. This is the Laplace-Runge-Lenz vector.

1.2 Solving using Hamilton's vector

Choose the initial conditions so that at time $t = 0$ the particle lies at perihelion, $r_{min} = b$, at$\varphi = 0$. This is a turning point, so $\dot{r} = 0$ and

$$\mathbf{v} = v_0\mathbf{j} = \mu b\dot{\varphi}_0\mathbf{j}$$

Then Hamilton's vector is

$$\begin{aligned}
\mathbf{h} &= \mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}} \\
&= \left(\mu b\dot{\varphi}_0 - \frac{\mu\alpha}{L}\right)\mathbf{j}
\end{aligned}$$

At any later time,

$$\begin{aligned}
(\mathbf{h} \cdot \hat{\boldsymbol{\varphi}})_{initial} &= \mathbf{h} \cdot \hat{\boldsymbol{\varphi}} \\
\left(\mu b\dot{\varphi}_0 - \frac{\mu\alpha}{L}\right)\mathbf{j} \cdot \hat{\boldsymbol{\varphi}} &= \left(\mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}\right) \cdot \hat{\boldsymbol{\varphi}} \\
\left(\mu b\dot{\varphi}_0 - \frac{\mu\alpha}{L}\right)\cos\varphi &= \mu r\dot{\varphi} - \frac{\mu\alpha}{L} \\
\left(\frac{L}{b} - \frac{\mu\alpha}{L}\right)\cos\varphi &= \frac{L}{r} - \frac{\mu\alpha}{L}
\end{aligned}$$

and therefore

$$\begin{aligned}
r &= \frac{L}{\frac{\mu\alpha}{L} + \left(\frac{L}{b} - \frac{\mu\alpha}{L}\right)\cos\varphi} \\
&= \frac{L^2}{\mu\alpha}\frac{1}{1 + \left(\frac{L^2}{\mu\alpha b} - 1\right)\cos\varphi}
\end{aligned}$$

1.3 Fitting the constants

So far, our solution is expressed in terms of constants L and r_0. It is convenient to define

$$\begin{aligned} r_0 &\equiv \frac{L^2}{\mu\alpha} \\ \varepsilon &\equiv \frac{L^2}{\mu\alpha b} - 1 \end{aligned}$$

so that the orbit equation takes the simpler form

$$r = \frac{r_0}{1 + \varepsilon \cos\varphi}$$

Then at $\varphi = 0$ and $\varphi = \pi$, r lies along the x axis, so the length of the semimajor axis is

$$\begin{aligned} 2a &= \frac{r_0}{1+\varepsilon} + \frac{r_0}{1-\varepsilon} \\ &= \frac{r_0 (1-\varepsilon) + r_0 (1+\varepsilon)}{1-\varepsilon^2} \\ &= \frac{2r_0}{1-\varepsilon^2} \\ a &= \frac{r_0}{1-\varepsilon^2} \end{aligned}$$

Along the y axis we have the *semi latus rectum*, that is, the distance to the ellipse from the center of force at the focus, perpendicular to the major axis,

$$\begin{aligned} 2p &= \frac{r_0}{1+\varepsilon\cos\frac{\pi}{2}} + \frac{r_0}{1+\varepsilon\cos\frac{3\pi}{2}} \\ 2p &= 2r_0 \\ p &= r_0 \end{aligned}$$

For $\varepsilon < 1$ we have p. The semiminor axis has length b equal to the maximum y-coordinate, where $y = r\sin\varphi$. Thus

$$\begin{aligned} y &= \frac{p\sin\varphi}{1+\varepsilon\cos\varphi} \\ 0 &= \frac{dy}{d\varphi} \\ &= \frac{p\cos\varphi}{1+\varepsilon\cos\varphi} - \frac{-\varepsilon p \sin^2\varphi}{(1+\varepsilon\cos\varphi)^2} \\ &= \frac{p\cos\varphi\,(1+\varepsilon\cos\varphi) + \varepsilon p\sin^2\varphi}{(1+\varepsilon\cos\varphi)^2} \\ &= \frac{p\cos\varphi + \varepsilon p\cos^2\varphi + \varepsilon p \sin^2\varphi}{(1+\varepsilon\cos\varphi)^2} \end{aligned}$$

Then

$$\cos\varphi_m = -\varepsilon$$

and therefore,

$$b = \frac{p\sin\varphi_m}{1+\varepsilon\cos\varphi_m}$$

$$\begin{aligned}
&= \frac{p\sqrt{1-\cos^2\varphi_m}}{1-\varepsilon^2} \\
&= p\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon^2} \\
&= a\sqrt{1-\varepsilon^2}
\end{aligned}$$

The energy is

$$\begin{aligned}
E &= \frac{L^2}{2\mu b^2} + V(b) \\
&= \frac{L^2}{2\mu b^2} - \frac{\alpha}{b}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{L^2}{2\mu b^2} - \frac{\alpha}{b} - E &= 0 \\
\frac{1}{b} &= \frac{\alpha \pm \sqrt{\alpha^2 + 2\frac{EL^2}{\mu}}}{\frac{L^2}{\mu}} \\
\frac{1}{b} &= \frac{\alpha\mu}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{\alpha^2\mu}}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varepsilon &\equiv \frac{L^2}{\mu\alpha b} - 1 \\
&= \sqrt{1 + \frac{2EL^2}{\alpha^2\mu}}
\end{aligned}$$

and we have the solution in terms of energy and angular momentum,

$$\begin{aligned}
r &= \frac{p}{1+\varepsilon\cos\varphi} \\
\varepsilon &= \sqrt{1 + \frac{2Ep}{\alpha}} \\
p &= \frac{L^2}{\mu\alpha} \\
a &= \frac{p}{1-\varepsilon^2} \\
b &= a\sqrt{1-\varepsilon^2}
\end{aligned}$$