Generalized coordinates

You are already familiar with using different coordinate systems to describe physical problems. You have used Cartesian, cylindrical and spherical coordinates for problems with those symmetries. But the number of possible coordinate systems is unlimited. As long as we can locate any point in space unambiguously, we may use any triple of numbers.

Consider an example in the plane. Consider the region above the diagonal line $y = x$ and to the right of the positive y-axis. In this region, let r be the radius of any circle, and consider the set of hyperbolas given by

$$y^2 - x^2 = \lambda^2$$

Each point in the given region has exactly one circle and one hyperbola passing through it. Therefore, specifying the pair (r, λ) uniquely determines any point in the region. To see this explicitly, we can find the (x, y) coordinates of the same point. Since $y > 0$, we have

$$y = +\sqrt{x^2 + \lambda^2}$$

We also know that

$$\begin{aligned} r &= +\sqrt{x^2 + y^2} \\ &= +\sqrt{2x^2 + \lambda^2} \end{aligned}$$

so that

$$2x^2 = r^2 - \lambda^2$$

and this is positive for any point in the region so we have

$$\begin{aligned} x &= +\sqrt{\frac{1}{2}\left(r^2 - \lambda^2\right)} \\ y &= +\sqrt{\frac{1}{2}\left(r^2 - \lambda^2\right) + \lambda^2} \\ &= +\sqrt{\frac{1}{2}\left(r^2 + \lambda^2\right)} \end{aligned}$$

and these values are unique throughout the region.

As a second example, consider the sinusoidal curves

$$y = \sin x + \alpha$$

together with the vertical lines

$$(x, y) = (\beta, y)$$

for all α, β. It is easy to see that each pair (α, β) determines exactly one point,

$$(x, y) = (\beta, \alpha + \sin\beta)$$

There are two important things to realize here:

1. *Any* set of parameters in continuous, one-to-one correspondence with points in space may serve as coordinates.

2. Our physical predictions must be independent of our choice of coordinates.

It follows from these points that we may choose coordinates that make our problem simpler to solve.

Constraints

Suppose we have a particle of mass m moving in 2-dimensions, constrained to move on the surface of the hyperbola

$$y^2 - x^2 = c^2$$

under a force of gravity, $-mg\hat{\mathbf{j}}$. We can find adapted coordinates to this problem. Let one coordinate be arc-length along the hyperbola, and the other be a series of hyperbolas,

$$v(x,y) = y^2 - x^2$$

To find length along the hyperbola, notice that the differential of the equation for the hyperbola gives

$$\begin{aligned} 2ydy - 2xdx &= 0 \\ \frac{dy}{dx} &= \frac{x}{y} \end{aligned}$$

while Pythagorean arc-length is

$$\begin{aligned} du &= \sqrt{dx^2 + dy^2} \\ u(x,y) &= \int \sqrt{dx^2 + dy^2} \\ &= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int \sqrt{1 + \left(\frac{x}{y}\right)^2} dx \\ &= \int^x \sqrt{1 + \left(\frac{x^2}{c^2 + x^2}\right)} dx \end{aligned}$$

The important thing to note here is that we can now specify the constraint on the motion by fixing one of the coordinates,

$$v = c^2$$

This means that the constraint is integrable.

In general, if we can express a constraint by a relationship between any set of coordinated, then it is integrable. There are constraints for which this is not possible, such as rolling, where the constraint is expressed as a relationship between not only the coordinates but also changes in the coordinates. It is useful (for many purposes!) to be able to decide when a set of equations is integrable.

Suppose a constraint can be expressed by some relationship between coordinates only. Then we may write this relationship as the vanishing of some function,

$$f(x_i) = 0$$

In the example above, this would be

$$\begin{aligned} f(x,y) &= y^2 - x^2 - c^2 \\ &= 0 \end{aligned}$$

Such a constraint always gives rise to a differential constraint,

$$df = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} dx_i = 0$$

where our system is described by N coordinates.

Conversely, if we have a differential constraint

$$\sum_i g_i dx_i = 0$$

it certianly remains true if we multiply by an arbitrary integrating factor, $h(x_i)$,

$$\sum_i hg_i dx_i = 0$$

Then there exists a function relating the coordinate if and only if

$$hg_i = \frac{\partial f}{\partial x_i}$$

for some function f. If we integrate this:

$$\begin{aligned}
\sum \frac{\partial f}{\partial x_i} dx_i &= \sum hg_i dx_i \\
\int df &= \sum hg_i dx_i \\
f &= \int \sum hg_i dx_i
\end{aligned}$$

To integrate the right side, we need to choose a path of integration. Let the path be a curve C specified by $x_i(\lambda)$. Then we have

$$f = \sum_i \int_C hg_i \frac{dx_i}{d\lambda} d\lambda$$

Now, f is a function if and only if, its value is independent of the path of integration. This is just like the question of the existence of a Newtonian potential – equality on all curves is equivalent to the condition that the integral vanish around every closed curve,

$$\oint_{C_1 - C_2} \left(\sum_i hg_i \frac{dx_i}{d\lambda} \right) d\lambda = 0$$

Thinking of the sum as a dot product in some N-dimensional space,

$$\mathbf{g} \equiv hg_i$$

we may apply Stoke's theorem,

$$\begin{aligned}
0 &= \oint_{C_1 - C_2} \left(\sum_i hg_i \frac{dx_i}{d\lambda} \right) d\lambda \\
&= \oint_{C_1 - C_2} \mathbf{g} \cdot \frac{d\mathbf{x}}{d\lambda} d\lambda \\
&= \iint_S (\nabla \times \mathbf{g}) \cdot \mathbf{n} d\lambda
\end{aligned}$$

and because of the freedom to choose $\mathbf{n}$ and the surface S, we must have

$$\nabla \times \mathbf{g} = 0$$

where the curl in higher dimensions is replaced by

$$\frac{\partial\left(hg_i\right)}{\partial x_j}-\frac{\partial\left(hg_j\right)}{\partial x_i}=0$$

This result holds in any number of dimensions, and is the necessary and sufficient condition for the integrability of the differential equation

$$\sum_i g_i dx_i = 0$$

This condition is the same as the equality of mixed partials of the function f. If we write out the components of the curl,

$$\frac{\partial\left(hg_i\right)}{\partial x_j}-\frac{\partial\left(hg_j\right)}{\partial x_i}=0$$

we see that this is just

$$\frac{\partial f}{\partial x_j \partial x_i}-\frac{\partial f}{\partial x_i \partial x_j}=0$$