Problem: Foucault Pendulum

A pendulum moves under the influence of gravity, suspended from a long cable with tension $\mathbf{T}$. The equation of motion is

$$\mathbf{T} - mg\mathbf{k} - 2m\boldsymbol{\omega} \times \mathbf{v} = m\mathbf{a}$$

where the centrifugal force makes only a slight change in the magnitude and direction of $\mathbf{g}$,

$$|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})| = \omega^2 r \cos\alpha$$

with direction away from the axis of rotation.

Let the angle of oscillation of the pendulum be very small (and the supporting cable very long) so that the arc of the pendulum is sufficiently close to horizontal motion. The net external force is the combined effect of the tension and gravity, equal to a restoring force, $mg\sin\theta$, so

$$mg\sin\theta \approx mg\theta = \frac{mg}{L}L\theta$$

giving an effective Hooke's law force with "spring" constant $\frac{mg}{L}$. The restoring force is therefore

$$\mathbf{T} - mg\mathbf{k} \approx -\frac{mg}{L}\rho\left(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi\right)$$

where $\rho = \sqrt{x^2+y^2}$ and motion in the $\mathbf{k}$ direction is negligible. The velocity and angular velocity at latitude α are

$$\begin{aligned}
\boldsymbol{\omega} &= \omega\left(\mathbf{j}\cos\alpha + \mathbf{k}\sin\alpha\right) \\
\mathbf{v} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j}
\end{aligned}$$

and the equation of motion becomes

$$\begin{aligned}
-\frac{mg}{L}\rho\left(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi\right) - 2m\omega\left(\mathbf{j}\cos\alpha + \mathbf{k}\sin\alpha\right) \times \left(\dot{x}\mathbf{i} + \dot{y}\mathbf{j}\right) &= m\left(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}\right) \\
-\frac{g}{L}x\mathbf{i} - \frac{g}{L}y\mathbf{j} + 2\omega\dot{x}\mathbf{k}\cos\alpha - 2\omega\dot{x}\mathbf{j}\sin\alpha + 2\omega\dot{y}\mathbf{i}\sin\alpha &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}
\end{aligned}$$

Equating like components,

$$\begin{aligned}
-\frac{g}{L}x\mathbf{i} + 2\omega\dot{y}\mathbf{i}\sin\alpha &= \ddot{x}\mathbf{i} \\
-\frac{g}{L}y\mathbf{j} - 2\omega\dot{x}\mathbf{j}\sin\alpha &= \ddot{y}\mathbf{j} \\
2\omega\dot{x}\mathbf{k}\cos\alpha &= \ddot{z}\mathbf{k}
\end{aligned}$$

The z-direction may be ignored since any acceleration in this direction acts only as a mild perturbation on the acceleration of gravity. We confirm later that $\omega\dot{x} \ll g$. Then we have

$$\begin{aligned}
\ddot{x} + \frac{g}{L}x &= 2\omega\sin\alpha\dot{y} \\
\ddot{y} + \frac{g}{L}y &= -2\omega\sin\alpha\dot{x}
\end{aligned}$$

We present three ways of solving this system of equations.

Method 1: Explicit rotating frame

We first solve this by putting in a slow rotation in the xy plane. Let

$$\begin{aligned}
x' &= x\cos\omega_p t - y\sin\omega_p t \\
y' &= x\sin\omega_p t + y\cos\omega_p t
\end{aligned}$$

Then:

$$\begin{aligned}
\dot{x}' &= \dot{x}\cos\omega_p t - x\omega_p \sin\omega_p t - \dot{y}\sin\omega_p t - y\omega_p\cos\omega_p t \\
&= (\dot{x} - y\omega_p)\cos\omega_p t - (x\omega_p + \dot{y})\sin\omega_p t \\
\dot{y}' &= \dot{x}\sin\omega_p t + x\omega_p\cos\omega_p t + \dot{y}\cos\omega_p t - y\omega_p\sin\omega_p t \\
&= (\dot{x} - y\omega_p)\sin\omega_p t + (x\omega_p + \dot{y})\cos\omega_p t
\end{aligned}$$

and

$$\begin{aligned}
\ddot{x}' &= (\ddot{x} - \dot{y}\omega_p)\cos\omega_p t - (\dot{x} - y\omega_p)\,\omega_p\sin\omega_p t - (\dot{x}\omega_p + \ddot{y})\sin\omega_p t - (x\omega_p + \dot{y})\,\omega_p\cos\omega_p t \\
&= \left(\ddot{x} - 2\dot{y}\omega_p - x\omega_p^2\right)\cos\omega_p t - \left(\ddot{y} + 2\dot{x}\omega_p - y\omega_p^2\right)\sin\omega_p t \\
\ddot{y}' &= (\ddot{x} - \dot{y}\omega_p)\sin\omega_p t + (\dot{x} - y\omega_p)\,\omega_p\cos\omega_p t + (\dot{x}\omega_p + \ddot{y})\cos\omega_p t - (x\omega_p + \dot{y})\,\omega_p\sin\omega_p t \\
&= \left(\ddot{x} - 2\dot{y}\omega_p - x\omega_p^2\right)\sin\omega_p t + \left(\ddot{y} + 2\dot{x}\omega_p - y\omega_p^2\right)\cos\omega_p t
\end{aligned}$$

Now substitute into the equations of motion,

$$\begin{aligned}
\ddot{x}' + \frac{g}{L}x' &= 2\omega\sin\alpha\dot{y}' \\
-\left(\ddot{y} + 2\dot{x}\omega_p - y\omega_p^2\right)\sin\omega_p t - \frac{g}{L}y\sin\omega_p t - (2\omega\sin\alpha\dot{x} - 2\omega\sin\alpha y\omega_p)\sin\omega_p t &= -\frac{g}{L}x\cos\omega_p t - \left(\ddot{x} - 2\dot{y}\omega_p - x\omega_p^2\right)\cos\omega_p t + \big(\\
-\left(\ddot{y} + \left(\frac{g}{L} - \omega_p^2 - 2\omega\sin\alpha\omega_p\right)y + 2\left(\omega\sin\alpha + \omega_p\right)\dot{x}\right)\sin\omega_p t &= -\left(\ddot{x} + \left(\frac{g}{L} - 2\omega\sin\alpha\omega_p - \omega_p^2\right)x - 2\big(\omega\sin\right. \\
\ddot{y}' + \frac{g}{L}y' &= -2\omega\sin\alpha\dot{x}' \\
\left(\ddot{y} + \left(\frac{g}{L} - \omega_p^2 - 2\omega\omega_p\sin\alpha\right)y + 2\left(\omega\sin\alpha + \omega_p\right)\dot{x}\right)\cos\omega_p t &= -\left(\ddot{x} + \left(\frac{g}{L} - 2\omega\sin\alpha\omega_p - \omega_p^2\right)x - 2\big(\omega\sin\right.
\end{aligned}$$

and therefore, both equations are solved if

$$\begin{aligned}
\ddot{y} + \left(\frac{g}{L} - 2\omega\sin\alpha\omega_p - \omega_p^2\right)y + 2\left(\omega\sin\alpha + \omega_p\right)\dot{x} &= 0 \\
\ddot{x} + \left(\frac{g}{L} - 2\omega\sin\alpha\omega_p - \omega_p^2\right)x - 2\left(\omega\sin\alpha + \omega_p\right)\dot{y} &= 0
\end{aligned}$$

which decouples when we choose ω_p to cancel the mixing term

$$\omega_p = -\omega\sin\alpha$$

In this rotation frame, the equations of motion reduce to simple oscillation for both x and y,

$$\begin{aligned}
\ddot{y} + \tilde{\omega}^2 y &= 0 \\
\ddot{x} + \tilde{\omega}^2 x &= 0
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\omega} &= \sqrt{\frac{g}{L} - 2\omega\omega_p\sin\alpha - \omega_p^2} \\
&= \sqrt{\frac{g}{L} + 2\left(\frac{\omega_P}{\sin\alpha}\right)\omega_p\sin\alpha - \omega_p^2} \\
&= \sqrt{\frac{g}{L} + \omega_p^2} \\
&\approx \sqrt{\frac{g}{L}}
\end{aligned}$$

Finally, we compare magnitudes. We assumed that $\omega\dot{x} \ll g$. But

$$\begin{aligned} x &= A\sin\tilde{\omega}t \\ \dot{x} &= A\tilde{\omega}\cos\tilde{\omega}t \end{aligned}$$

where $\dot{x} \sim A\tilde{\omega} = L\theta_{max}\tilde{\omega} \ll L\tilde{\omega}$. Therefore,

$$\begin{aligned} \omega\dot{x} &\ll L\tilde{\omega}\omega \\ &= L\sqrt{\frac{g}{L}}\omega \end{aligned}$$

Since we know the rotation rate of Earth is much less than the pendulum frequency, $\omega \ll \tilde{\omega}$ this gives

$$\omega\dot{x} \ll g$$

and our assumption is justified.

Method 2: Coriolis theorem

We may simplify this calculation by making a second use of the Coriolis theorem, changing to a rotating frame in the xy-plane

First, write our equations of motion,

$$\begin{aligned} m\ddot{x} &= -\frac{mg}{L}x + 2m\omega\sin\alpha\dot{y} \\ m\ddot{y} &= -\frac{mg}{L}y - 2m\omega\sin\alpha\dot{x} \end{aligned}$$

as a vector equation. Let $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ so that

$$m\ddot{\mathbf{x}} = -\frac{mg}{L}\mathbf{x} - 2m\left(\omega\sin\alpha\mathbf{k}\right)\times\dot{\mathbf{x}}$$

This shows that we have a 2-dimensional system with an effective external force

$$\mathbf{F}_{eff} = -\frac{mg}{L}\mathbf{x} - 2m\left(\omega\sin\alpha\mathbf{k}\right)\times\dot{\mathbf{x}}$$

Now rewrite this equation in a frame of reference rotating with constant angular velocity $\boldsymbol{\omega}_p = \omega_p\mathbf{k}$. Then the (full) Coriolis theorem gives the equation of motion as

$$\begin{aligned} m\ddot{\mathbf{x}}' &= \mathbf{F}_{eff} - 2m\boldsymbol{\omega}_p\times\dot{\mathbf{x}}' - m\boldsymbol{\omega}_p\times\left(\boldsymbol{\omega}_p\times\mathbf{x}'\right) \\ &= -\frac{mg}{L}\mathbf{x} - 2m\left(\omega\sin\alpha\mathbf{k}\right)\times\dot{\mathbf{x}}' - 2m\boldsymbol{\omega}_p\times\dot{\mathbf{x}}' - m\omega_p^2\mathbf{x}' \end{aligned}$$

so if we set

$$\boldsymbol{\omega}_p = -\omega\sin\alpha\mathbf{k}$$

the cross product terms cancel and we have

$$\ddot{\mathbf{x}}' = -\frac{g}{L}\mathbf{x}' - \omega_p^2\mathbf{x}'$$

$$\ddot{\mathbf{x}} + \left(\frac{g}{L} + \omega_p^2\right)\mathbf{x} = 0$$

which is the equation of a simple oscillator with frequency

$$\omega = \sqrt{\frac{g}{L} + \omega_p^2}$$

in agreement with the previous result.

Method 3: Complex variables We may also define

$$z = x + iy$$

so adding i times the second equation to the first, the pair of equations becomes the single complex equation

$$\begin{aligned} \ddot{z} + 2i\omega_P \dot{z} + \frac{g}{L} z &= 0 \\ \omega_P &= \omega \sin\alpha \end{aligned}$$

Let w be a variable rotating with angular velocity ω_P relative to z, i.e., $z = we^{-i\omega_P t}$. Then

$$\begin{aligned} \left(\ddot{w} - 2i\omega_p \dot{w} - \omega_P^2 w\right) + 2i\omega_P \left(\dot{w} - i\omega_P w\right) + \frac{g}{L} w &= 0 \\ \ddot{w} + \left(\frac{g}{L} + \omega_P^2\right) w &= 0 \end{aligned}$$

so that w moves with the simple oscillation

$$\begin{aligned} w &= A \sin\omega t + B \cos\omega t \\ \omega &= \sqrt{\frac{g}{L} + \omega_P^2} \end{aligned}$$

Each of the three methods shows that in a reference frame rotating with angular velocity $\omega_P = -\omega \sin\alpha$, there is simple harmonic motion of the pendulum with frequency $\sqrt{\frac{g}{L} + \omega_P^2}$.