Atwood's machine and the raindrop

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1 Atwood's machine with massive rope

The same as problem 1, but now let the rope have non-negligible mass. Let the total length of the rope be $L = l + \pi R$ and have mass m. Assume $m + m_1 < m_2$ so the motion is in the same direction as in problem 1. Let the initial position be such that m_2 even with the center of the pulley so that m_1 is a distance l below the center of the pulley. Find the acceleration $a(t)$. Notice that then tension in the rope is not constant. Show that after simplification, the equations of motion for the system can be reduced to a single equation of the form

$$\alpha + \beta x = \frac{a}{g}$$

where x is the length of rope on the m_2 side of the pulley, and

$$\begin{aligned}
M_{eff} &= \frac{1}{2}M + m_2 + m_1 + m \\
\alpha &= \frac{1}{M_{eff}}\left(m_2 - m_1 - \frac{ml}{L}\right) \\
\beta &= \frac{2m}{LM_{eff}}
\end{aligned}$$

Integrate this equation to find $x(t)$.

Let x be a clockwise coordinate along the rope, starting at the initial position of m_2. We will need (with $\lambda = 1$, or set $\lambda = 0$ to neglect the segment of rope on the pulley,

$$\begin{aligned}
I &= \frac{1}{2}MR^2 + \lambda\left[\frac{(\pi R)}{L}m\right]R^2 \\
\tau &= I\alpha \\
\tau &= -T_1'R + T_2'R \\
\alpha &= \frac{a}{R} \\
F_1 &= -m_1 g + T_1 \\
F_2 &= m_2 g - T_2 \\
f_{r1} &= -T_1 + T_1' - mg\frac{l-x}{L} \\
f_{r2} &= T_2' - T_2 + mg\frac{x}{L} \\
L &= l + \pi R
\end{aligned}$$

so then we have five equations of motion,

$$-T_1'R + T_2'R = \left(\frac{1}{2}MR^2 + \lambda\left[\frac{(\pi R)}{L}m\right]R^2\right)\left(\frac{a}{R}\right)$$

$$
\begin{aligned}
-m_1 g + T_1 &= m_1 a \\
m_2 g - T_2 &= m_2 a \\
T_1^{'} - T_1 - mg\frac{l-x}{L} &= m\frac{l-x}{L}a \\
-T_2^{'} + T_2 + \frac{mg}{L}x &= \frac{m}{L}xa
\end{aligned}
$$

where we have intermediate unknowns $T_1, T_1^{'}, T_2, T_2^{'}$.

Simplify

$$
\begin{aligned}
T_2^{'} - T_1^{'} &= \left(\frac{1}{2}M + \lambda\left[\frac{(\pi R)}{L}m\right]\right)a \\
T_1 &= m_1\left(g+a\right) \\
T_2 &= m_2\left(g-a\right) \\
T_1^{'} - T_1 &= \frac{m}{L}\left(l-x\right)\left(g+a\right) \\
T_2^{'} - T_2 &= \frac{m}{L}x\left(g-a\right)
\end{aligned}
$$

Eliminate T_1 and T_2. With

$$
\begin{aligned}
T_1 &= m_1\left(g+a\right) \\
T_2 &= m_2\left(g-a\right)
\end{aligned}
$$

the remaining equations become

$$
\begin{aligned}
T_2^{'} - T_1^{'} &= \left(\frac{1}{2}M + \lambda\left[\frac{(\pi R)}{L}m\right]\right)a \\
T_1^{'} &= \left(\frac{m}{L}\left(l-x\right) + m_1\right)\left(g+a\right) \\
T_2^{'} &= \left(\frac{m}{L}x + m_2\right)\left(g-a\right)
\end{aligned}
$$

This gives the solutions for T_1' and T_2' as

$$
\begin{aligned}
T_1^{'} &= \left(\frac{m}{L}\left(l-x\right) + m_1\right)\left(g+a\right) \\
T_2^{'} &= \left(\frac{m}{L}x + m_2\right)\left(g-a\right)
\end{aligned}
$$

Substituting into the final equation gives

$$
\begin{aligned}
\left(\frac{m}{L}x + m_2\right)\left(g-a\right) - \left(\frac{m}{L}\left(l-x\right) + m_1\right)\left(g+a\right) &= \left(\frac{1}{2}M + \lambda\left[\frac{(\pi R)}{L}m\right]\right)a \\
\left(\frac{m}{L}x + m_2\right)g - \left(\frac{m}{L}\left(l-x\right) + m_1\right)g - \left(\frac{m}{L}x + m_2\right)a - \left(\frac{m}{L}\left(l-x\right) + m_1\right)a &= \left(\frac{1}{2}M + \lambda\left[\frac{(\pi R)}{L}m\right]\right)a \\
\left(\frac{m}{L}x + m_2 - \frac{m}{L}\left(l-x\right) - m_1\right)g &= \left(\frac{1}{2}M + \lambda\left[\frac{(\pi R)}{L}m\right] + \frac{m}{L}x + m_2 + \frac{m}{L}\left(l-\right.\right. \\
\left(m_2 - \frac{m}{L}\left(l-2x\right) - m_1\right)g &= \left(\frac{1}{2}M + \lambda\left[\frac{(\pi R)}{L}m\right] + m_2 + \frac{ml}{L} + m_1\right)a \\
\left(m_2 - \frac{m}{L}\left(l-2x\right) - m_1\right)g &= \left(\frac{1}{2}M + m_2 + m_1 + m\right)a
\end{aligned}
$$

This result makes perfect sense. On the left, the driving force is the difference between the gravitational force on the masses together with the difference between the gravitational force on the straight sections of rope. On the right, we have the sum of every inertial term: the moment of inertia of the pulley, the two masses, and the full mass of the rope.

Now, integrate. To find the energy, multiply by $\dot{x}$. Letting the initial position and velocity be $x = 0, \dot{x} = 0$,

$$\begin{aligned}
\left(m_2 - \frac{m}{L}(l - 2x) - m_1\right) g \frac{dx}{dt} &= \left(\frac{1}{2}M + m_2 + m_1 + m\right) \dot{x} \frac{d\dot{x}}{dt} \\
\int_0^x \left(m_2 - \frac{m}{L}(l - 2x) - m_1\right) g dx &= \int_0^{\dot{x}} \left(\frac{1}{2}M + m_2 + m_1 + m\right) \dot{x} d\dot{x} \\
\left(m_2 - m_1 - \frac{m}{L} l\right) gx + \frac{mg}{L} x^2 &= \frac{1}{2}\left(\frac{1}{2}M + m_2 + m_1 + m\right) \dot{x}^2
\end{aligned}$$

To simplify, define

$$\begin{aligned}
\mathcal{M} &\equiv \frac{1}{2}M + m_2 + m_1 + m \\
\mathfrak{m} &\equiv m_2 - m_1 - \frac{m}{L} l > 0
\end{aligned}$$

where the inequality must hold if the acceleration is to be positive. We have

$$\begin{aligned}
\mathfrak{m} gx + \frac{mg}{L} x^2 &= \frac{1}{2}\mathcal{M}\dot{x}^2 \\
\sqrt{\frac{2}{\mathcal{M}}\left(\mathfrak{m} gx + \frac{mg}{L} x^2\right)} &= \frac{dx}{dt} \\
\int_0^t dt &= \int_0^x \frac{dx}{\sqrt{\frac{2g\mathfrak{m}}{\mathcal{M}}\left(x + \frac{m}{\mathfrak{m}L} x^2\right)}} \\
\sqrt{\frac{2gm}{\mathcal{M}L}} \int_0^t dt &= \int_0^x \frac{dx}{\sqrt{\left(\frac{\mathfrak{m}L}{m} x + x^2\right)}} \\
\sqrt{\frac{2gm}{\mathcal{M}L}} t &= \int_0^x \frac{dx}{\sqrt{\frac{\mathfrak{m}L}{m} x + x^2}}
\end{aligned}$$

Now, completing the square,

$$x^2 + \frac{\mathfrak{m}L}{m} x = \left(x + \frac{1}{2}\frac{\mathfrak{m}L}{m}\right)^2 - \left(\frac{\mathfrak{m}L}{2m}\right)^2$$

Then letting $z = x + \frac{1}{2}\frac{\mathfrak{m}L}{m}$, we need

$$\sqrt{\frac{2gm}{\mathcal{M}L}} t = \int_0^x \frac{dz}{\sqrt{z^2 - \left(\frac{\mathfrak{m}L}{2m}\right)^2}}$$

Now let $z = \left(\frac{\mathfrak{m}L}{2m}\right)\cosh\lambda$ so that

$$\begin{aligned}
\sqrt{\frac{2gm}{\mathcal{M}L}} t &= \int_0^x \frac{\left(\frac{\mathfrak{m}L}{2m}\right)\sinh\lambda d\lambda}{\left(\frac{\mathfrak{m}L}{2m}\right)\sinh\lambda} \\
\sqrt{\frac{2gm}{\mathcal{M}L}} t &= \lambda|_0^x
\end{aligned}$$

$$
\begin{aligned}
\sqrt{\frac{2gm}{\mathcal{M}L}}t &= \cosh^{-1} \left.\frac{2mz}{\mathfrak{m}L}\right|_0^x \\
\sqrt{\frac{2gm}{\mathcal{M}L}}t &= \cosh^{-1} \frac{2m\left(x+\frac{1}{2}\frac{\mathfrak{m}L}{m}\right)}{\mathfrak{m}L} - \cosh^{-1} 1 \\
\sqrt{\frac{2gm}{\mathcal{M}L}}t &= \cosh^{-1} \frac{2m\left(x+\frac{1}{2}\frac{\mathfrak{m}L}{m}\right)}{\mathfrak{m}L} \\
\cosh\sqrt{\frac{2gm}{\mathcal{M}L}}t &= \frac{2m}{\mathfrak{m}L}x+1
\end{aligned}
$$

and finally

$$
x = \frac{\mathfrak{m}L}{2m}\left(\cosh\sqrt{\frac{2gm}{\mathcal{M}L}}t - 1\right)
$$

Different initial conditions

Using

$$
\begin{aligned}
\cosh(a)\cosh(b) - \sinh(a)\sinh(b) &= \frac{1}{4}\left[\left(e^a+e^{-a}\right)\left(e^b+e^{-b}\right) - \left(e^a-e^{-a}\right)\left(e^b-e^{-b}\right)\right] \\
&= \frac{1}{4}\left[e^ae^b+e^ae^{-b}+e^{-a}e^b+e^{-a}e^{-b}-e^ae^b+e^ae^{-b}+e^{-a}e^b-e^{-a}e^{-b}\right] \\
&= \frac{1}{2}\left(e^ae^{-b}+e^ae^{-b}\right) \\
&= \cosh(a+b)
\end{aligned}
$$

we have

$$
\begin{aligned}
x &= \frac{\mathfrak{m}L}{2m}\left(\cosh\alpha\left(t-t_0\right)-1\right) \\
&= \frac{\mathfrak{m}L}{2m}\left(\cosh(\alpha t)\cosh(\alpha t_0) - \sinh(\alpha t)\sinh(\alpha t_0) - 1\right)
\end{aligned}
$$

which is of the form

$$
x = \frac{\mathfrak{m}L}{2m}\left(A\cosh(\alpha t) - B\sinh(\alpha t) - 1\right)
$$

so we might get any linear combination of $e^{\alpha t}$ and $e^{-\alpha t}$, depending on our choice of initial conditions.

2 Raindrop in fog

Consider an initially infinitesimally small droplet of water falls from rest in a uniform fog of density $\rho < 1$, under the influence of gravity, mg. As the drop falls, it sweeps up water from the fog and grows in size, and hence mass, so that the drop sweeps out a conical section of fog. By solving completely for $x(t)$, show that the acceleration of the drop is constant and equal to $a = \frac{1}{7}g$.

This problem has two essential degrees of freedom, but four possible independent variables: m, r, x and t. The first step is to eliminate two of these in favor of the remaining two. Any pair should work, though some choices may seem easier. The second step is to find an integrating factor.

Let x be measured positive downward from the initial position at time $t = 0$. The drop sweeps up mass ($\rho_W = 1$) in proportion to its cross-sectional area:

$$
\begin{aligned}
F &= mg \\
dm &= \pi\rho r^2 dx
\end{aligned}
$$

$$
\begin{aligned}
m &= \frac{4}{3}\pi\rho_W r^3 \\
dm &= 4\pi\rho_W r^2 dr \\
\pi\rho r^2 dx &= 4\pi\rho_W r^2 dr \\
dx &= \frac{4\rho_W}{\rho} dr
\end{aligned}
$$

$$
\begin{aligned}
\frac{dm}{m} &= \frac{\pi r^2 \rho dx}{\frac{4}{3}\pi\rho_W r^3} \\
&= \frac{3\rho}{4\rho_W r} dx \\
&= \frac{3\rho}{4\rho_W r}\frac{4\rho_W}{\rho} dr \\
&= \frac{3}{r} dr
\end{aligned}
$$

Then

$$
\begin{aligned}
mg &= \frac{d}{dt}\left(m\dot{x}\right) \\
g &= \frac{1}{m}\left(\dot{m}\dot{x} + m\ddot{x}\right) \\
g &= \frac{\dot{m}}{m}\dot{x} + \ddot{x} \\
g &= \left(\frac{3}{r}\dot{r}\right)\left(\frac{4\rho_W}{\rho}\dot{r}\right) + \frac{4\rho_W}{\rho}\ddot{r} \\
\frac{g\rho}{4\rho_W} &= \frac{3}{r}\dot{r}^2 + \ddot{r}
\end{aligned}
$$

Multiply by r^3,

$$
\begin{aligned}
r^3\frac{g\rho}{4\rho_W} &= 3r^2\dot{r}^2 + r^3\ddot{r} \\
r^3\frac{g\rho}{4\rho_W} &= (\dot{r^3\dot{r}})
\end{aligned}
$$

Now multiply by $r^3\dot{r}$ and integrate,

$$
\begin{aligned}
r^3\frac{g\rho}{4\rho_W} &= (\dot{r^3\dot{r}}) \\
\frac{g\rho}{4\rho_W} r^6 dr &= \left(r^3\dot{r}\right) d(\dot{r^3\dot{r}}) \\
\int_0^r \frac{g\rho}{4\rho_W} r^6 dr &= \int_0^{r^3\dot{r}} \left(r^3\dot{r}\right) d(\dot{r^3\dot{r}}) \\
\frac{g\rho}{4\rho_W}\frac{r^7}{7} &= \frac{1}{2}\left(r^3\dot{r}\right)^2 \\
\sqrt{\frac{g\rho}{14\rho_W}} r dt &= dr
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\frac{g\rho}{14\rho_W}}t &= \frac{dr}{\sqrt{r}} \\
\sqrt{\frac{g\rho}{14\rho_W}}t &= 2\sqrt{r} \\
\frac{g\rho}{14\rho_W}t^2 &= 4r \\
\frac{1}{2}\left(\frac{g}{7}\right)t^2 &= \frac{4\rho_W}{\rho}r \\
\frac{1}{2}\left(\frac{g}{7}\right)t^2 &= x
\end{aligned}
$$

So finally, we have solved for $x(t)$,

$$x(t) = \frac{1}{2}\left(\frac{g}{7}\right)t^2$$

showing that the drop falls with constant acceleration, $\frac{g}{7}$.