Differential forms

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1 From differentials to differential forms

In a formal sense, we may define differentials as the vector space of linear mappings from curves to the reals, that is, given a differential df we may use it to map any curve, $C \in \mathcal{C}$ to a real number simply by integrating:

$$\begin{aligned} df : \mathcal{C} &\rightarrow R \\ x &= \int\limits_C df \end{aligned}$$

This suggests a generalization, since we know how to integrate over surfaces and volumes as well as curves. In higher dimensions we also have higher order multiple integrals. We now consider the integrands of arbitrary multiple integrals

$$\int f(\mathbf{x})dl, \int\int f(\mathbf{x})dS, \int\int\int f(\mathbf{x})dV \tag{1}$$

Much of their importance lies in the coordinate invariance of the resulting integrals.

One of the important properties of integrands is that they can all be regarded as oriented. If we integrate a line integral along a curve from A to B we get a number, while if we integrate from B to A we get minus the same number,

$$\int_A^B f(\mathbf{x})dl = -\int_B^A f(\mathbf{x})dl \tag{2}$$

We can also demand oriented surface integrals, so the surface integral

$$\int\int \mathbf{A} \cdot \mathbf{n}\, dS \tag{3}$$

changes sign if we reverse the direction of the normal to the surface. This normal can be thought of as the cross product of two basis vectors within the surface. If these basis vectors' cross product is taken in one order, $\mathbf{n}$ has one sign. If the opposite order is taken then $-\mathbf{n}$ results. Similarly, volume integrals change sign if we change from a right- or left-handed coordinate system.

The generalization from differentials to differential forms, and the associated vector calculus makes use of three operations: the wedge product, the exterior derivative, and the Hodge dual. We discuss this in turn.

1.1 The wedge product

We can build this alternating sign into our convention for writing differential forms by introducing a formal antisymmetric product, called the *wedge* product, symbolized by $\wedge$, which is defined to give these differential elements the proper signs. Thus, surface integrals will be written as integrals over the products

$$\mathbf{d}x \wedge \mathbf{d}y, \mathbf{d}y \wedge \mathbf{d}z, \mathbf{d}z \wedge \mathbf{d}x$$

with the convention that $\wedge$ is antisymmetric:

$$\mathbf{d}x \wedge \mathbf{d}y = -\mathbf{d}y \wedge \mathbf{d}x$$

under the interchange of any two basis forms. This automatically gives the right orientation of the surface. Similarly, the volume element becomes

$$\mathbf{V} = \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$$

which changes sign if any pair of the basis elements are switched.

We can go further than this by formalizing the full integrand. For a line integral, the general form of the integrand is a linear combination of the basis differentials,

$$A_x \mathbf{d}x + A_y \mathbf{d}y + A_z \mathbf{d}z$$

Notice that we simply add the different parts. Similarly, a general surface integrand is

$$A_z \mathbf{d}x \wedge \mathbf{d}y + A_y \mathbf{d}z \wedge \mathbf{d}x + A_x \mathbf{d}y \wedge \mathbf{d}z$$

while the volume integrand is

$$f(x)\, \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$$

These objects are called *differential forms.*

Clearly, differential forms come in several types. Functions are called 0 -forms, line elements 1-forms, surface elements 2-forms, and volume forms are called 3-forms. These are all the types that exist in 3 -dimensions, but in more than three dimensions we can have p-forms with p ranging from zero to the dimension, d, of the space. Since we can take arbitrary linear combinations of p-forms, they form a vector space, Λ_p.

We can always wedge together any two forms. We assume this wedge product is associative, and obeys the usual distributive laws. The wedge product of a p-form with a q-form is a $(p+q)$-form.

Notice that the antisymmetry is all we need to rearrange any combination of forms. In general, wedge products of even order forms with any other forms commute while wedge products of pairs of odd-order forms anticommute. In particular, functions (0-forms) commute with all p-forms. Using this, we may interchange the order of a line element and a surface area, for if

$$\begin{aligned}
\mathbf{l} &= A\mathbf{d}x \\
\mathbf{S} &= B\mathbf{d}y \wedge \mathbf{d}z
\end{aligned}$$

then

$$\begin{aligned}
\mathbf{l} \wedge \mathbf{S} &= (A\ \mathbf{d}x) \wedge (B\ \mathbf{d}y \wedge \mathbf{d}z) \\
&= A\ \mathbf{d}x \wedge B\ \mathbf{d}y \wedge \mathbf{d}z \\
&= AB\ \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z \\
&= -AB\ \mathbf{d}y \wedge \mathbf{d}x \wedge \mathbf{d}z \\
&= AB\ \mathbf{d}y \wedge \mathbf{d}z \wedge \mathbf{d}x \\
&= \mathbf{S} \wedge \mathbf{l}
\end{aligned}$$

but the wedge product of two line elements changes sign, for it

$$\begin{aligned}
\mathbf{l}_1 &= A\mathbf{d}x \\
\mathbf{l}_2 &= B\mathbf{d}y + C\mathbf{d}z
\end{aligned}$$

then

$$
\begin{aligned}
\mathbf{l}_1 \wedge \mathbf{l}_2 &= (A\ \mathbf{d}x) \wedge (B\mathbf{d}y + C\mathbf{d}z) \\
&= A\ \mathbf{d}x \wedge B\mathbf{d}y + A\ \mathbf{d}x \wedge C\mathbf{d}z \\
&= AB\ \mathbf{d}x \wedge \mathbf{d}y + AC\ \mathbf{d}x \wedge \mathbf{d}z \\
&= -AB\ \mathbf{d}y \wedge \mathbf{d}x - AC\ \mathbf{d}z \wedge \mathbf{d}x \\
&= -B\mathbf{d}y \wedge A\mathbf{d}x - C\mathbf{d}z \wedge A\mathbf{d}x \\
&= -\mathbf{l}_2 \wedge \mathbf{l}_1
\end{aligned} \tag{4}
$$

For any odd-order form, ω, we immediately have

$$\omega \wedge \omega = -\omega \wedge \omega = 0$$

In 3-dimensions there are no 4-forms because anything we try to construct must contain a repeated basis form. For example

$$
\begin{aligned}
\mathbf{l} \wedge \mathbf{V} &= (A\ \mathbf{d}x) \wedge (B\ \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z) \\
&= AB\ \mathbf{d}x \wedge \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z \\
&= 0
\end{aligned}
$$

since $\mathbf{d}x \wedge \mathbf{d}x = 0$. The same occurs for anything we try. Of course, if we have more dimensions then there are more independent directions and we can find nonzero 4-forms. In general, in d-dimensions we can find d-forms, but no $(d+1)$-forms.

Now suppose we want to change coordinates. How does an integrand change? Suppose Cartesian coordinates (x, y) in the plane are given as some functions of new coordinates (u, v). Then we already know that differentials change according to

$$\mathbf{d}x = \mathbf{d}x\,(u, v) = \frac{\partial x}{\partial u}\mathbf{d}u + \frac{\partial x}{\partial v}\mathbf{d}v$$

and similarly for $\mathbf{d}y$, applying the usual rules for partial differentiation. Notice what happens when we use the wedge product to calculate the new area element:

$$
\begin{aligned}
\mathbf{d}x \wedge \mathbf{d}y &= \left(\frac{\partial x}{\partial u}\mathbf{d}u + \frac{\partial x}{\partial v}\mathbf{d}v\right) \wedge \left(\frac{\partial y}{\partial u}\mathbf{d}u + \frac{\partial y}{\partial v}\mathbf{d}v\right) \\
&= \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\mathbf{d}v \wedge \mathbf{d}u + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\mathbf{d}u \wedge \mathbf{d}v \\
&= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathbf{d}u \wedge \mathbf{d}v \\
&= \mathcal{J}\ \mathbf{d}u \wedge \mathbf{d}v
\end{aligned}
$$

where

$$\mathcal{J} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is the Jacobian of the coordinate transformation. This is exactly the way that an area element changes when we change coordinates! Notice the Jacobian coming out automatically. We couldn't ask for more – the wedge product not only gives us the right signs for oriented areas and volumes, but gives us the right transformation to new coordinates. Of course the volume change works, too.

In eq.(4), showing the anticommutation of two 1-forms, identify the property of form multiplication used in each step (associativity, anticommutation of basis forms, commutation of 0-forms, etc.).

Show that under a coordinate transformation

$$\begin{aligned} x &\rightarrow x(u,v,w) \\ y &\rightarrow y(u,v,w) \\ z &\rightarrow z(u,v,w) \end{aligned}$$

the new volume element is just get the full Jacobian times the new volume form,

$$\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z = \mathcal{J}(xyz; uvw)\ \mathbf{d}u \wedge \mathbf{d}v \wedge \mathbf{d}w$$

So the wedge product successfully keeps track of p-dim volumes and their orientations in a coordinate invariant way. Now any time we have an integral, we can regard the integrand as being a differential form. But all of this can go much further. Recall our proof that 1-forms form a vector space. Thus, the differential, $\mathbf{d}x$, of $x(u,v)$ given above is just a gradient. It vanishes along surfaces where x is constant, and the components of the vector

$$\left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right)$$

point in a direction normal to those surfaces. So symbols like $\mathbf{d}x$ or $\mathbf{d}u$ contain directional information. Writing them with a boldface $\mathbf{d}$ indicates this vector character. Thus, we write

$$\mathbf{A} = A_i \mathbf{d}x^i$$

Let

$$f(x,y) = axy$$

Show that the vector with components

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

is perpendicular to the surfaces of constant f.

Let's sum up. We have defined forms, have written down their formal properties, and have use those properties to write them in components. Then, we defined the wedge product, which enables us to write p-dimensional integrands as p-forms in such a way that the orientation and coordinate transformation properties of the integrals emerges automatically.

Though it is 1-forms, $A_i \mathbf{d}x^i$ that correspond to vectors, we have defined a product of basis forms that we can generalize to more complicated objects. Many of these objects are already familiar. Consider the product of two 1-forms.

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B} &= A_i \mathbf{d}x^i \wedge B_j \mathbf{d}x^j \\ &= A_i B_j \mathbf{d}x^i \wedge \mathbf{d}x^j \\ &= \frac{1}{2} A_i B_j \left(\mathbf{d}x^i \wedge \mathbf{d}x^j - \mathbf{d}x^j \wedge \mathbf{d}x^i\right) \\ &= \frac{1}{2}\left(A_i B_j \mathbf{d}x^i \wedge \mathbf{d}x^j - A_i B_j \mathbf{d}x^j \wedge \mathbf{d}x^i\right) \\ &= \frac{1}{2}\left(A_i B_j \mathbf{d}x^i \wedge \mathbf{d}x^j - A_j B_i \mathbf{d}x^i \wedge \mathbf{d}x^j\right) \\ &= \frac{1}{2}\left(A_i B_j - A_j B_i\right) \mathbf{d}x^i \wedge \mathbf{d}x^j \end{aligned}$$

The coefficients

$$A_i B_j - A_j B_i$$

are essentially the components of the cross product. We will see this in more detail below when we discuss the curl.

1.2 The exterior derivative

We may regard the differential of any function, say $f(x, y, z)$, as the 1-form:

$$\begin{aligned}\mathbf{d}f &= \frac{\partial f}{\partial x}\mathbf{d}x + \frac{\partial f}{\partial y}\mathbf{d}y + \frac{\partial f}{\partial z}\mathbf{d}z \\ &= \frac{\partial f}{\partial x^i}\mathbf{d}x^i\end{aligned}$$

Since a function is a 0-form then we can imagine an operator $\mathbf{d}$ that differentiates any 0-form to give a 1-form. In Cartesian coordinates, the coefficients of this 1-form are just the Cartesian components of the gradient.

The operator $\mathbf{d}$ is called the *exterior derivative*, and we may apply it to any p-form to get a $(p+1)$-form. The extension is defined as follows. First consider a 1-form

$$\mathbf{A} = A_i \mathbf{d}x^i$$

We define

$$\mathbf{dA} = \mathbf{d}A_i \wedge \mathbf{d}x^i$$

Similarly, since an arbitrary p-form in n-dimensions may be written as

$$\omega = A_{i_1 i_2 \cdots i_p}\mathbf{d}x^{i_1} \wedge \quad \mathbf{d}x^{i_2} \cdots \wedge \mathbf{d}x^{i_p}$$

we define the exterior derivative of ω to be the $(p+1)$-form

$$\mathbf{d}\omega = \mathbf{d}A_{i_1 i_2 \cdots i_p} \wedge \mathbf{d}x^{i_1} \wedge \mathbf{d}x^{i_2} \cdots \wedge \mathbf{d}x^{i_p}$$

Let's see what happens if we apply $\mathbf{d}$ twice to the Cartesian coordinate, x, regarded as a function of x, y and z:

$$\begin{aligned}\mathbf{d}^2 x &= \mathbf{d}(\mathbf{d}x) \\ &= \mathbf{d}(1\mathbf{d}x) \\ &= \mathbf{d}(1) \wedge \mathbf{d}x \\ &= 0\end{aligned}$$

since all derivatives of the constant function $f = 1$ are zero. The same applies if we apply $\mathbf{d}$ twice to *any* function:

$$\begin{aligned}\mathbf{d}^2 f &= \mathbf{d}(\mathbf{d}f) \\ &= \mathbf{d}\left(\frac{\partial f}{\partial x^i}\mathbf{d}x^i\right) \\ &= \mathbf{d}\left(\frac{\partial f}{\partial x^i}\right) \wedge \mathbf{d}x^i \\ &= \left(\frac{\partial^2 f}{\partial x^j \partial x^i}\mathbf{d}x^j\right) \wedge \mathbf{d}x^i \\ &= \frac{\partial^2 f}{\partial x^j \partial x^i}\mathbf{d}x^j \wedge \mathbf{d}x^i\end{aligned}$$

By the same argument we used to get the components of the curl, we may write this as

$$\begin{aligned}\mathbf{d}^2 f &= \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j}\right)\mathbf{d}x^j \wedge \mathbf{d}x^i \\ &= 0\end{aligned}$$

since partial derivatives commute.

Prove the Poincaré Lemma: $\mathbf{d}^2\omega = 0$ where ω is an arbitrary p-form.

Next, consider the effect of $\mathbf{d}$ on an arbitrary 1-form. We have

$$\begin{aligned}\mathbf{dA} &= \mathbf{d}\left(A_i \mathbf{d}x^i\right) \\ &= \left(\frac{\partial A_i}{\partial x^j}\mathbf{d}x^j\right) \wedge \mathbf{d}x^i \\ &= \frac{1}{2}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\mathbf{d}x^j \wedge \mathbf{d}x^i \qquad (5)\end{aligned}$$

We have the components of the curl of the vector $\mathbf{A}$. We must be careful here, however, because these are the components of the curl only in Cartesian coordinates. Later we will see how these components relate to those in a general coordinate system. Also, recall from Section (4.2.2) that the components A_i are distinct from the usual vector components A^i. These differences will be resolved when we give a detailed discussion of the metric in Section (5.6). Ultimately, the action of $\mathbf{d}$ on a 1-form gives us a coordinate invariant way to calculate the curl.

Finally, suppose we have a 2-form expressed as

$$\mathbf{S} = A_z \mathbf{d}x \wedge \mathbf{d}y + A_y \mathbf{d}z \wedge \mathbf{d}x + A_x \mathbf{d}y \wedge \mathbf{d}z$$

Then applying the exterior derivative gives

$$\begin{aligned}\mathbf{dS} &= \mathbf{d}A_z \wedge \mathbf{d}x \wedge \mathbf{d}y + \mathbf{d}A_y \wedge \mathbf{d}z \wedge \mathbf{d}x + \mathbf{d}A_x \wedge \mathbf{d}y \wedge \mathbf{d}z \\ &= \frac{\partial A_z}{\partial z}\mathbf{d}z \wedge \mathbf{d}x \wedge \mathbf{d}y + \frac{\partial A_y}{\partial y}\mathbf{d}y \wedge \mathbf{d}z \wedge \mathbf{d}x + \frac{\partial A_x}{\partial x}\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z \\ &= \left(\frac{\partial A_z}{\partial z} + \frac{\partial A_y}{\partial y} + \frac{\partial A_x}{\partial x}\right)\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z \qquad (6)\end{aligned}$$

so that the exterior derivative can also reproduce the divergence.

Fill in the missing steps in the derivation of eq.(6).

Compute the exterior derivative of the arbitrary 3-form, $\mathbf{A} = f(x, y, z)\,\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$.

1.3 The Hodge dual

To truly have the curl in eq.(6) or the curl in eq.(5), we need a way to turn a 2-form into a vector, i.e., a 1-form and a way to turn a 3-form into a 0-form. This leads us to introduce the Hodge dual, or star, operator, *.

Notice that in 3-dim, both 1-forms and 2-forms have three independent components, while both 0- and 3-forms have one component. This suggests that we can define an invertible mapping between these pairs. In Cartesian coordinates, suppose we set

$$\begin{aligned}{}^*\left(\mathbf{d}x \wedge \mathbf{d}y\right) &= \mathbf{d}z \\ {}^*\left(\mathbf{d}y \wedge \mathbf{d}z\right) &= \mathbf{d}x \\ {}^*\left(\mathbf{d}z \wedge \mathbf{d}x\right) &= \mathbf{d}y \\ {}^*\left(\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z\right) &= 1\end{aligned}$$

and further require the star to be its own inverse,

$$^{**} = 1$$

With these rules we can find the Hodge dual of any form in 3-dim.

Show that the dual of a general 1-form,

$$\mathbf{A} = A_i \mathbf{d}x^i$$

is the 2-form

$$\mathbf{S} = A_z \mathbf{d}x \wedge \mathbf{d}y + A_y \mathbf{d}z \wedge \mathbf{d}x + A_x \mathbf{d}y \wedge \mathbf{d}z$$

Show that for an arbitrary (Cartesian) 1-form

$$\mathbf{A} = A_i \mathbf{d}x^i$$

that

$$^*\mathbf{d}^*\mathbf{A} = divA$$

Write the curl of **A**

$$curl\,(A) = \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)\mathbf{d}x + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{d}y + \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\mathbf{d}z$$

in terms of the exterior derivative and the Hodge dual.

Write the Cartesian dot product of two 1-forms in terms of wedge products and duals.

We have now shown how three operations – the wedge product $\wedge$, the exterior derivative $\mathbf{d}$, and the Hodge dual * – together encompass the usual dot and cross products as well as the divergence, curl and gradient. In fact, they do much more – they extend all of these operations to arbitrary coordinates and arbitrary numbers of dimensions. To explore these generalizations, we must first explore properties of the metric and look at coordinate transformations. This will allow us to define the Hodge dual in arbitrary coordinates.

2 Transformations and the volume form

2.1 Transformations

Since the use of orthonormal frames is simply a convenient choice of basis, no information is lost in restricting our attention to them. We can always return to general frames if we wish. But as long as we maintain the restriction, we can work with a reduced form of the symmetry group. Arbitrary coordinate transformations – diffeomorphisms – preserve the class of frames, but only orthogonal transformations preserve orthonormal frames. Nonetheless, the class of tensors is remains unchanged – there is a 1-1, onto correspondence between tensors with diffeomorphism covariance and those with orthogonal covariance.

The correspondence between general frame and orthonormal frame tensors is provided by the orthonormal frame itself. Given an orthonormal frame

$$\mathbf{e}^a = e_m{}^a \mathbf{d}x^m$$

we can use the coefficient matrix $e_m{}^a$ and its inverse to transform back and forth between orthonormal and coordinate indices. Thus, given any vector in an arbitrary coordinate basis,

$$\mathbf{v} = v^m \frac{\partial}{\partial x^m}$$

we may insert the identity in the form

$$\delta_n^m = e_n{}^a e_a{}^m$$

to write

$$\begin{aligned}
\mathbf{v} &= v^n \delta_n^m \frac{\partial}{\partial x^m} \\
&= v^n e_n{}^a e_a{}^m \frac{\partial}{\partial x^m} \\
&= \left(v^n e_n{}^a\right) \mathbf{e}_a \\
&= v^a \mathbf{e}_a
\end{aligned}$$

The mapping

$$v^a = v^n e_n{}^a$$

is invertible because $e_n{}^a$ is invertible. Similarly, any tensor, for example

$$T^{m_1 \ldots m_r}{}_{n_1 \ldots n_s}$$

may be written in an orthonormal basis by using one factor of $e_m{}^a$ or $e_a{}^n$ for each linear slot:

$$T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = T^{m_1 \ldots m_r}{}_{n_1 \ldots n_s} e_{m_1}{}^{a_1} \ldots e_{m_n}{}^{a_r} e_{b_1}{}^{n_1} \ldots e_{b_s}{}^{n_s}$$

Similar expressions may be written for tensors with their contravariant and covariant indices in other orders.

We showed in Section (3) that the components of the metric are related to the Cartesian components by

$$g_{jk} = \frac{\partial x^m}{\partial y^j} \frac{\partial x^n}{\partial y^k} \eta_{mn}$$

where we have corrected the index positions and inserted the Cartesian form of the metric explicitly as $\eta_{mn} = diag(1,1,1)$. Derive the form of the metric in cylindrical coordinates directly from the coordinate transformation,

$$\begin{aligned}
x &= x(\rho, \varphi, z) = \rho \cos \varphi \\
y &= y(\rho, \varphi, z) = \rho \sin \varphi \\
z &= z(\rho, \varphi, z) = z
\end{aligned}$$

Notice that the identity matrix should exist in any coordinate system, since multiplying any vector by the identity should be independent of coordinate system. Show that the matrix $\delta^i{}_j$, defined to be the unit matrix in one coordinate system, has the same form in every other coordinate system. Notice that the upper index will transform like a contravariant vector and the lower index like a covariant vector. Also note that $\delta^i{}_j = \delta_j{}^i$.

Show that the inverse to the metric transforms as a contravariant second rank tensor. The easiest way to do this is to use the equation

$$g_{ij} g^{jk} = \delta_i^k$$

and the result of exercise 2, together with the transformation law for g_{ij}.

2.2 The volume form

So far, we have only defined the Levi-Civita tensor in Cartesian coordinates, where it is given by the totally antisymmetric symbol

$$\varepsilon_{i_1 i_2 \ldots i_n}$$

in n dimensions. This symbol, however, is not quite a tensor because under a diffeomorphism it becomes

$$\varepsilon_{i_1 i_2 \ldots i_n} \frac{\partial x^{i_1}}{\partial y^{j_1}} \frac{\partial x^{i_2}}{\partial y^{j_2}} \ldots \frac{\partial x^{i_n}}{\partial y^{j_n}} = J \varepsilon_{j_1 j_2 \ldots j_n}$$

where

$$J = \det \left(\frac{\partial x^m}{\partial y^n} \right)$$

is the Jacobian of the coordinate transformation. The transformation is linear and homogeneous, but J is a density not a scalar. We can correct for this to form a tensor by dividing by another density. The most convenient choice is the determinant of the metric. Since the metric transforms as

$$g'_{mn} = \frac{\partial x^i}{\partial y^m} \frac{\partial x^j}{\partial y^n} g_{ij}$$

the determinants are related by

$$\begin{aligned} g' &= \det g'_{mn} \\ &= \det \left(\frac{\partial x^i}{\partial y^m} g_{ij} \frac{\partial x^j}{\partial y^n} \right) \\ &= \det \frac{\partial x^i}{\partial y^m} \deg t_{ij} \det \frac{\partial x^j}{\partial y^n} \\ &= J^2 g \end{aligned}$$

Therefore, in the combination

$$e_{i \ldots j} = \sqrt{g} \varepsilon_{i \ldots j}$$

the factors of J cancel, leaving

$$e'_{i \ldots j} = \sqrt{g'} \varepsilon_{i \ldots j}$$

so that $e_{i \ldots j}$ is a tensor. If we raise all indices on $e_{i_1 i_2 \ldots i_n}$, using n copies of the inverse metric, we have

$$\begin{aligned} e^{j_1 j_2 \cdots j_n} &= \sqrt{g} g^{j_1 i_1} g^{j_2 i_2} \ldots g^{j_n i_n} \varepsilon_{i_1 i_2 \ldots i_n} \\ &= \sqrt{g} g^{-1} \varepsilon^{j_1 j_2 \ldots j_n} \\ &= \frac{1}{\sqrt{g}} \varepsilon^{j_1 j_2 \ldots j_n} \end{aligned}$$

This is also a tensor.

3 Calculus using differential forms

Define a p-form as a linear map from oriented p-dimensional volumes to the reals:

$$\Lambda_p : V_p \to R$$

Linearity refers to both the forms and the volumes. Thus, for any two p -forms, Λ_p^1 and Λ_p^2, and any constants a and b,

$$a\Lambda_p^1 + b\Lambda_p^2$$

is also a p-form, while for any two disjoint p-volumes, V_p^1 and V_p^2,

$$\Lambda_p \left(V_p^1 + V_p^2 \right) = \Lambda_p \left(V_p^1 \right) + \Lambda_p \left(V_p^2 \right)$$

In Section 3, we showed for 1-forms that these conditions specify the differential of functions. For p-forms, they are equivalent to linear combinations of wedge products of p 1-forms.

Let $\mathbf{A}$ be a p-form in d-dimensions. Then, inserting a convenient normalization,

$$\mathbf{A} = \frac{1}{p!} A_{i_1 \ldots i_p} \mathbf{d} x^{i_1} \wedge \ldots \wedge \mathbf{d} x^{i_p}$$

The action of the exterior derivative, $\mathbf{d}$, on such a p-form is

$$\mathbf{dA} = \frac{1}{p!}\left(\frac{\partial}{\partial x^k} A_{i_1 \ldots i_p}\right) \mathbf{d}x^k \wedge \mathbf{d}x^{i_1} \wedge \ldots \wedge \mathbf{d}x^{i_p}$$

We also defined the wedge product as a distributive, associative, antisymmetric product on 1-forms:

$$\begin{aligned}
\left(a\mathbf{d}x^i + b\mathbf{d}x^i\right) \wedge \mathbf{d}x^j &= a\mathbf{d}x^i \wedge \mathbf{d}x^j + b\mathbf{d}x^i \wedge \mathbf{d}x^j \\
\mathbf{d}x^i \wedge \left(\mathbf{d}x^j \wedge \mathbf{d}x^k\right) &= \left(\mathbf{d}x^i \wedge \mathbf{d}x^j\right) \wedge \mathbf{d}x^k \\
\mathbf{d}x^i \wedge \mathbf{d}x^j &= -\mathbf{d}x^j \wedge \mathbf{d}x^i
\end{aligned}$$

A third operation, the Hodge dual, was provisionally defined in Cartesian coordinates, but now we can write its full definition. The dual of $\mathbf{A}$ is defined to be the $(d-p)$-form

$$^*\mathbf{A} = \frac{1}{(d-p)!p!} A_{i_1 \ldots i_p} e^{i_1 \ldots i_p}{}_{i_{p+1} \ldots i_d} \mathbf{d}x^{i_{p+1}} \wedge \ldots \wedge \mathbf{d}x^{i_d}$$

Notice that we have written the first p indices of the Levi-Civita tensor in the superscript position to keep with our convention of always summing an up index with a down index. In Cartesian coordinates, these two forms represent the same array of numbers, but it makes a difference when we look at other coordinate systems.

Differential calculus is defined in terms of these three operations, $(\wedge, ^*, \mathbf{d})$. Together, they allow us to perform all standard calculus operations in any number of dimensions and in a way independent of any coordinate choice.

3.1 Grad, Div, Curl and Laplacian

It is straightforward to write down the familiar operations of gradient and curl and divergence. We specify each, and apply each in polar coordinates, (ρ, θ, z). Recall that the metric in polar coordinates is

$$g_{mn} = \begin{pmatrix} 1 & & \\ & \rho^2 & \\ & & 1 \end{pmatrix}$$

its inverse is

$$g^{mn} = \begin{pmatrix} 1 & & \\ & \frac{1}{\rho^2} & \\ & & 1 \end{pmatrix}$$

and its determinant is

$$g = \det g_{mn} = \rho^2$$

Gradient The gradient of a function is given by the exterior derivative of a 0 -form,

$$\mathbf{d}f = \frac{\partial f}{\partial x^i} \mathbf{d}x^i$$

Notice that the coefficients are components of a type-$\binom{0}{1}$ tensor, so that if we want the gradient to be a vector, we require the metric:

$$[\nabla f]^i = g^{ij} \frac{\partial f}{\partial x^j}$$

For example, the gradient in polar coordinates has components

$$[\nabla f]^i = \begin{pmatrix} 1 & & \\ & \frac{1}{\rho^2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \varphi} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \frac{1}{\rho}\frac{\partial f}{\partial \varphi} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

so

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

Divergence The use of differential forms leads to an extremely useful expression for the divergence – important enough that it goes by the name of the divergence theorem. Starting with a 1-form, $\omega = \omega_i \mathbf{d}x^i$, we compute

$$\begin{aligned}
{}^*\mathbf{d}{}^*\omega &= {}^*\mathbf{d}{}^*\omega_i \mathbf{d}x^i \\
&= {}^*\mathbf{d}\left(\frac{1}{2}\omega_i e^i{}_{jk}\right)\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2}{}^*\mathbf{d}\left(\omega_i \sqrt{g} g^{in}\right)\varepsilon_{njk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2}{}^*\frac{\partial}{\partial x^m}\left(\omega_i \sqrt{g} g^{in}\right)\varepsilon_{njk}\mathbf{d}x^m \wedge \mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2}\frac{\partial}{\partial x^m}\left(\omega_i \sqrt{g} g^{in}\right)\varepsilon_{njk}e^{mjk} \\
&= \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}\left(\omega_i \sqrt{g} g^{in}\right)\varepsilon_{njk}\varepsilon^{mjk} \\
&= \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}\left(\omega_i \sqrt{g} g^{in}\right)2\delta^m_n \\
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}\left(\omega_i \sqrt{g} g^{im}\right)
\end{aligned}$$

In terms of the vector, rather than form, components of the original form, we may replace $\omega^i = g^{ij}\omega_j$ so that

$${}^*\mathbf{d}{}^*\omega = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}\left(\sqrt{g}\omega^m\right) = \nabla \cdot \omega$$

Since the operations on the left are all coordinate invariant, the in the middle is also. Notice that in Cartesian coordinates the metric is just δ_{ij}, with determinant 3, so the expression reduces to the familiar form of the divergence and

$$\nabla \cdot \omega = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}\left(\sqrt{g}\omega^m\right)$$

In polar coordinates we have

$$\begin{aligned}
\nabla \cdot \omega &= \frac{1}{\sqrt{\rho^2}}\frac{\partial}{\partial x^m}\left(\sqrt{\rho^2}\omega^m\right) \\
&= \frac{1}{\sqrt{\rho^2}}\left(\frac{\partial}{\partial \rho}\left(\sqrt{\rho^2}\omega^\rho\right) + \frac{\partial}{\partial \varphi}\left(\sqrt{\rho^2}\omega^\varphi\right) + \frac{\partial}{\partial z}\left(\sqrt{\rho^2}\omega^z\right)\right) \\
&= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\omega^\rho\right) + \frac{\partial \omega^\varphi}{\partial \varphi} + \frac{\partial \omega^z}{\partial z}
\end{aligned}$$

Curl The curl is the dual of the exterior derivative of a 1-form. Thus, if $\omega = \omega_i \mathbf{d}x^i$ then

$$\begin{aligned}
{}^*\mathbf{d}\omega &= {}^*\frac{\partial}{\partial x_j}\omega_i \mathbf{d}x^j \mathbf{d}x^i \\
&= \left(e^{ji}{}_k \frac{\partial}{\partial x_j}\omega_i\right)\mathbf{d}x^k \\
&= e^{ji}{}_k g_{im} g^{mn} \frac{\partial}{\partial x_j}\omega_n \mathbf{d}x^k \\
&= e^{ji}{}_k g_{im}\left(\frac{\partial}{\partial x_j}\left(g^{mn}\omega_n\right) - \omega_n \frac{\partial}{\partial x_j} g^{mn}\right)\mathbf{d}x^k \\
&= e_{lmk} g^{lj}\left(\frac{\partial}{\partial x_j}\omega^m - \omega^s g_{sn}\frac{\partial}{\partial x_j}g^{mn}\right)\mathbf{d}x^k
\end{aligned}$$

Now observe that

$$\begin{aligned}
g_{sn}\frac{\partial}{\partial x_j}g^{mn} &= \frac{\partial}{\partial x_j}\left(g_{sn}g^{mn}\right) - g^{mn}\frac{\partial}{\partial x_j}\left(g_{sn}\right) \\
&= \frac{\partial}{\partial x_j}\delta^m_s - g^{mn}\frac{\partial}{\partial x_j}\left(g_{sn}\right) \\
&= -g^{mn}\frac{\partial}{\partial x_j}g_{sn}
\end{aligned}$$

so that

$$\begin{aligned}
{}^*\mathbf{d}\omega &= e_{lmk}g^{lj}\left(\frac{\partial}{\partial x_j}\omega^m + \omega^s g^{mn}\frac{\partial}{\partial x_j}g_{sn}\right)\mathbf{d}x^k \\
&= \left(e_{lmk}g^{lj}\frac{\partial}{\partial x_j}\omega^m + \omega^s e^{jn}{}_k \frac{\partial}{\partial x_j}g_{sn}\right)\mathbf{d}x^k
\end{aligned}$$

Next consider

$$\begin{aligned}
e^{jn}{}_k\frac{\partial}{\partial x_j}g_{sn} &= e^{jn}{}_k \partial_j g_{sn} \\
&= \frac{1}{2}e^{jn}{}_k\left(\partial_j g_{sn} - \partial_n g_{sj}\right) \\
&= \frac{1}{2}e^{jn}{}_k\left(\partial_j g_{sn} - \partial_n g_{sj} + \partial_s g_{jn}\right) \\
&= e^{jn}{}_k \Gamma_{nsj}
\end{aligned}$$

This combines to

$$\begin{aligned}
{}^*\mathbf{d}\omega &= \left(e_{lmk}g^{lj}\frac{\partial}{\partial x_j}\omega^m + \omega^s e^{jn}{}_k\frac{\partial}{\partial x_j}g_{sn}\right)\mathbf{d}x^k \\
&= \left(e_{lmk}g^{lj}\frac{\partial}{\partial x_j}\omega^m + \omega^s e^{jn}{}_k\Gamma_{nsj}\right)\mathbf{d}x^k \\
&= e^j_{mk}\left(\frac{\partial}{\partial x_j}\omega^m + g^{nm}\omega^s\Gamma_{nsj}\right)\mathbf{d}x^k \\
&= e^j_{mk}\left(\partial_j\omega^m + \omega^s\Gamma^m{}_{sj}\right)\mathbf{d}x^k \\
&= e^j_{mk}D_j\omega^m\mathbf{d}x^k \\
&= \left(e_{jmk}D^j\omega^m\right)\mathbf{d}x^k
\end{aligned}$$

Therefore, if we raise the free index, the curl is

$$\begin{aligned} [\nabla\times\omega]^i &= g^{ik}\left(e_{jmk}D^j\omega^m\right) \\ &= \frac{1}{\sqrt{g}}\varepsilon^{ijk}D_j\omega_k \end{aligned}$$

Also consider

$$\begin{aligned} \mathbf{d}^*\omega &= \mathbf{d}\left(e^i{}_{jk}\omega_i\mathbf{d}x^j\mathbf{d}x^k\right) \\ &= \mathbf{d}\left(e_{ijk}\omega^i\mathbf{d}x^j\mathbf{d}x^k\right) \\ &= \mathbf{d}\left(\sqrt{g}\varepsilon_{ijk}\omega^i\mathbf{d}x^j\mathbf{d}x^k\right) \\ &= \frac{\partial}{\partial x^m}\left(\sqrt{g}\omega^i\varepsilon_{ijk}\mathbf{d}x^m\mathbf{d}x^j\mathbf{d}x^k\right) \\ &= \left(e^{ji}{}_k\frac{\partial}{\partial x^j}\omega_i\right)\mathbf{d}x^k \end{aligned}$$

The simplest form computationally uses this to write

$$^*\mathbf{d}\omega = [\nabla\times\omega]^i g_{ik}\mathbf{d}x^k$$

To apply the formula, start with the components of the vector. In our familiar example in polar coordinates, let

$$w^i = (w^\rho, w^\varphi, w^z)$$

The corresponding form has components $\omega_i = g_{ij}w^j = \left(w^\rho, \rho^2 w^\varphi, w^z\right)$. Therefore, the exterior derivative is

$$\begin{aligned} \mathbf{d}\omega &= \mathbf{d}\left(w^\rho\mathbf{d}\rho + \rho^2 w^\varphi\mathbf{d}\varphi + w^z\mathbf{d}z\right) \\ &= \frac{\partial w^\rho}{\partial\varphi}\mathbf{d}\varphi\wedge\mathbf{d}\rho + \frac{\partial w^\rho}{\partial z}\mathbf{d}z\wedge\mathbf{d}\rho \\ &\quad +\frac{\partial}{\partial\rho}\left(\rho^2 w^\varphi\right)\mathbf{d}\rho\wedge\mathbf{d}\varphi + \frac{\partial}{\partial z}\left(\rho^2 w^\varphi\right)\mathbf{d}z\wedge\mathbf{d}\varphi \\ &\quad +\frac{\partial w^z}{\partial\rho}\mathbf{d}\rho\wedge\mathbf{d}z + \frac{\partial w^z}{\partial\varphi}\mathbf{d}\varphi\wedge\mathbf{d}z \\ &= \left(\frac{\partial}{\partial\rho}\left(\rho^2 w^\varphi\right) - \frac{\partial w^\rho}{\partial\varphi}\right)\mathbf{d}\rho\wedge\mathbf{d}\varphi + \left(\frac{\partial w^z}{\partial\varphi} - \frac{\partial}{\partial z}\left(\rho^2 w^\varphi\right)\right)\mathbf{d}\varphi\wedge\mathbf{d}z \\ &\quad +\left(\frac{\partial w^\rho}{\partial z} - \frac{\partial w^z}{\partial\rho}\right)\mathbf{d}z\wedge\mathbf{d}\rho \end{aligned}$$

Now the dual maps the basis as

$$\begin{aligned} {}^*\mathbf{d}\rho\wedge\mathbf{d}\varphi &= e^{123}g_{33}\mathbf{d}z = \frac{1}{\rho}\mathbf{d}z \\ {}^*\mathbf{d}\varphi\wedge\mathbf{d}z &= e^{231}g_{11}\mathbf{d}\rho = \frac{1}{\rho}\mathbf{d}\rho \\ {}^*\mathbf{d}z\wedge\mathbf{d}\rho &= e^{312}g_{22}\mathbf{d}\varphi = \rho\mathbf{d}\varphi \end{aligned}$$

so that

$$\begin{aligned} {}^*\mathbf{d}\omega &= \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\rho^2 w^\varphi\right) - \frac{\partial w^\rho}{\partial\varphi}\right)\mathbf{d}z + \left(\frac{1}{\rho}\frac{\partial w^z}{\partial\varphi} - \rho\frac{\partial}{\partial z}\left(w^\varphi\right)\right)\mathbf{d}\rho \\ &\quad +\rho\left(\frac{\partial w^\rho}{\partial z} - \frac{\partial w^z}{\partial\rho}\right)\mathbf{d}\varphi \end{aligned}$$

Now, since

$$^{*}\mathbf{d}\omega = [\nabla \times \omega]^{i}\, g_{ik}\mathbf{d}x^{k}$$

we use the inverse metric on the components of $^{*}\mathbf{d}\omega$ to find

$$\omega^{i} = g^{ij}\omega_{j}$$

so we have

$$\begin{aligned}
[\nabla \times \omega]^{1} &= \frac{1}{\rho}\frac{\partial w^{z}}{\partial \varphi} - \rho\frac{\partial}{\partial z}\left(w^{\varphi}\right) \\
[\nabla \times \omega]^{2} &= \frac{1}{\rho}\left(\frac{\partial \omega^{\rho}}{\partial z} - \frac{\partial \omega^{z}}{\partial \rho}\right) \\
[\nabla \times \omega]^{3} &= \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho^{2} w^{\varphi}\right) - \frac{\partial w^{\rho}}{\partial \varphi}\right)
\end{aligned}$$

Work out the form of the gradient, curl, divergence and laplacian in spherical coordinates. Express your results using a basis of unit vectors.

4 The Poincaré lemma and Stokes' theorem

We have already seen the Poincaré lemma,

$$\mathbf{d}^{2}\boldsymbol{\omega} \equiv 0$$

for any p-form $\boldsymbol{\omega}$. The extremely important Stokes' theorem is the converse, which states that for any p-form $\boldsymbol{\omega}$, if

$$\mathbf{d}\boldsymbol{\omega} = 0$$

throughout a concave region, then there exists a $(p-1)$-form $\boldsymbol{\eta}$ such that

$$\boldsymbol{\omega} = \mathbf{d}\boldsymbol{\eta}$$

This provides a converse because if $\boldsymbol{\omega} = \mathbf{d}\boldsymbol{\eta}$ then the Poincaré lemma immediately implies $\mathbf{d}\boldsymbol{\omega} = 0$.

Consider how this relates to the familiar 3-dim form of Stokes' theorem. If we restrict to 3-dim and integrate $\mathbf{d}\boldsymbol{\eta}$ along any curve the result depends only on the value of $\boldsymbol{\eta}$ at the endpoints,

$$\int_{C} \mathbf{d}\boldsymbol{\eta} = \boldsymbol{\eta}\left(\mathbf{x}_{2}\right) - \boldsymbol{\eta}\left(\mathbf{x}_{1}\right)$$

Therefore, around any closed curve the integral vanishes.

$$\oint_{C} \mathbf{d}\boldsymbol{\eta} = 0$$

Now consider the integral along a curve C for an arbitrary 1-form $\boldsymbol{\omega}$,

$$\eta = \int_{C} \boldsymbol{\omega}$$

This integral gives rise to a (single valued) function if and only if it vanishes on all closed paths,

$$\oint_{C} \boldsymbol{\omega} = 0$$

Stokes' theorem allows us to write this as a surface integral of $\mathbf{d}\boldsymbol{\omega}$,

$$\begin{aligned} 0 &= \oint_C \boldsymbol{\omega} \\ &= \iint_S \mathbf{d}\boldsymbol{\omega} \end{aligned}$$

and the arbitrariness of the surface S allows us to conclude

$$\mathbf{d}\boldsymbol{\omega} = 0$$

Show that if $\boldsymbol{\omega}$ is a 2-form in 3-dim, then the generalized Stokes' theorem is the divergence theorem.

5 Exercises: Maxwell's equations

In an orthonormal vector basis the electric and magnetic fields and the current are

$$\begin{aligned} \mathbf{E} &= E^i \mathbf{e}_i \\ \mathbf{B} &= B^i \mathbf{e}_i \\ \mathbf{J} &= J^i \mathbf{e}_i \end{aligned}$$

Define equivalent forms in arbitrary coordinates by

$$\begin{aligned} \epsilon &= E^i g_{ij} \mathbf{d}x^j = E_j \mathbf{d}x^j \\ \beta &= \frac{1}{2} B^i e_{ijk} \mathbf{d}x^j \wedge \mathbf{d}x^k \\ \kappa &= J^i g_{ij} \mathbf{d}x^j \end{aligned}$$

Show that Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{4\pi}{c}\rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \end{aligned}$$

may be written in terms of ϵ, β, κ and ρ as

$$\begin{aligned} {}^*\mathbf{d}^*\epsilon &= \frac{4\pi}{c}\rho \\ \mathbf{d}\beta &= 0 \\ \mathbf{d}\varepsilon + \frac{1}{c}\frac{\partial \beta}{\partial t} &= 0 \\ {}^*\mathbf{d}^*\beta - \frac{1}{c}\frac{\partial \epsilon}{\partial t} &= \frac{4\pi}{c}\kappa \end{aligned}$$

The third equation may be proved as follows:

$$\mathbf{d}\varepsilon + \frac{1}{c}\frac{\partial \beta}{\partial t} = \mathbf{d}\left(E^i g_{ij}\right)\mathbf{d}x^j + \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{2} B^i e_{ijk} \mathbf{d}x^j \wedge \mathbf{d}x^k$$

$$
\begin{aligned}
&= \frac{\partial E_j}{\partial x^m}\mathbf{d}x^m \wedge \mathbf{d}x^j + \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{2}B^i e_{ijk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^m}E_j - \frac{\partial}{\partial x^j}E_m\right)\mathbf{d}x^m \wedge \mathbf{d}x^j + \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{2}B^i e_{ijk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{4}\left(\frac{\partial}{\partial x^n}E_l - \frac{\partial}{\partial x^l}E_n\right)e^{inl}e_{ijk}\mathbf{d}x^j \wedge \mathbf{d}x^k + \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{2}B^i e_{ijk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial}{\partial x^n}E_l - \frac{\partial}{\partial x^l}E_n\right)e^{inl} + \frac{1}{c}\frac{\partial}{\partial t}B^i\right)e_{ijk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2}\left(e^{inl}\partial_n E_l + \frac{1}{c}\frac{\partial}{\partial t}B^i\right)e_{ijk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2}\left(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}\right)^i e_{ijk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= 0
\end{aligned}
$$

From Maxwell's equations,

$$
\begin{aligned}
{}^*\mathbf{d}^*\epsilon &= \frac{4\pi}{c}\rho \\
\mathbf{d}\beta &= 0 \\
\mathbf{d}\varepsilon + \frac{1}{c}\frac{\partial\beta}{\partial t} &= 0 \\
{}^*\mathbf{d}^*\beta - \frac{1}{c}\frac{\partial\ \epsilon}{\partial t} &= \frac{4\pi}{c}\kappa
\end{aligned}
$$

show that

$$
\frac{1}{c}\frac{\partial}{\partial t}\rho +^* \mathbf{d}^*\ \kappa = 0
$$

Show that this equation is the continuity equation by writing it in the usual vector notation.

Using the homogeneous Maxwell equations

$$
\begin{aligned}
\mathbf{d}\beta &= 0 \\
\mathbf{d}\varepsilon + \frac{1}{c}\frac{\partial\beta}{\partial t} &= 0
\end{aligned}
$$

show that the electric and magnetic fields arise from a potential.

Start with the magnetic equation

$$
\mathbf{d}\beta = 0
$$

Then the converse to the Poincaré lemma shows immediately that

$$
\beta = \mathbf{dA}
$$

for some 1-form $\mathbf{A}$. Substitute this result into the remaining homogeneous equation,

$$
\begin{aligned}
\mathbf{d}\varepsilon + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{dA} &= 0 \\
\mathbf{d}\left(\varepsilon + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}\right) &= 0
\end{aligned}
$$

A second use of the converse to the Poincaré lemma shows that there exist a 0-form $-\varphi$ such that

$$
\varepsilon + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} = -\mathbf{d}\varphi
$$

and therefore

$$
\varepsilon = -\mathbf{d}\varphi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}
$$