General coordinate covariance of the Euler Lagrange equations

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Here we show that the Euler-Lagrange equation is *covariant* under general coordinate transformations. By this we mean that if the Euler-Lagrange equation

$$V_i(x) \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

is satisfied in one set of coordinates, x^i, then it will hold in any other, y^i,

$$V_i(y) \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{y}^i} - \frac{\partial L}{\partial y^i} = 0$$

where $x^i\left(y^j\right)$ is the invertible coordinate transformation. For the two vectors to vanish together requires there to be a linear map from one to other, i.e., there exists some $J_i^{\ j}$ such that $V_i = \sum_j J_i^{\ j} V_j$, or

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i}\right) = \sum_{j=1}^{N} J_i^{\ j}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^j} - \frac{\partial L}{\partial y^j}\right)$$

It is clear what $J_i^{\ j}$ must be – if L is independent of velocity, we require

$$\frac{\partial L}{\partial x^i} = \sum_{j=1}^{N} J_i^{\ j}\frac{\partial L}{\partial y^j}$$

but the chain rule tells us that

$$\frac{\partial L}{\partial x^i} = \sum_{j=1}^{N} \frac{\partial y^j}{\partial x^i}\frac{\partial L}{\partial y^j}$$

Therefore, $J_i^{\ j}$ is the Jacobian matrix of the coordinate transformation, $\frac{\partial y^j}{\partial x^i}$. In conclusion, the Euler-Lagrangian equation hold in any coordinate system if and only if

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \sum_{j=1}^{N} \frac{\partial y^j}{\partial x^i}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^j} - \frac{\partial L}{\partial y^j}\right)$$

for any two, $x^i \to y^i$.

We prove that this is the case by deriving the relationship between the Euler-Lagrange equation for $x^i(t)$ and the Euler-Lagrange equation for $y^i(t)$.

Consider the variational equation for y^i, computed in two ways. Since the action may be written as either $S\left[x^i\right]$ or $S\left[y^i\right]$, we have

$$S\left[y^i\right] = S\left[x^i\left(y^k\right)\right]$$

First, we may immediately write the Euler-Lagrange equation by varying $S\left[y^i(t)\right]$. Following the usual steps, integrating by parts, we have

$$\begin{aligned}
\delta S &= \delta \int_C L\left(y^i, \dot{y}^i, t\right) dt \\
&= \sum_{k=1}^{N} \int_C \left(\frac{\partial L}{\partial y^k}\delta y^k + \frac{\partial L}{\partial \dot{y}^k}\delta \dot{y}^k\right) dt \\
&= \sum_{k=1}^{N} \int_C \left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k}\right)\right) \delta y^k dt
\end{aligned}$$

where the surface term vanishes in the final step because the variation is taken to vanish at the endpoints. Now compare what we get by varying $S\left[x^i\left(y^k\right)\right]$ with respect to $y^i(t)$:

$$\begin{aligned}
0 &= \delta S \\
&= \delta \int_C L\left(x^i\left(y^k, t\right), \dot{x}^i\left(y^k, \dot{y}^k, t\right)\right) dt \\
&= \sum_{i,k=1}^{N} \int_C \left(\frac{\partial L}{\partial x^k}\left(\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial x^k}{\partial \dot{y}^i}\delta \dot{y}^i\right) + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{\partial \dot{x}^k}{\partial y^i}\delta y^i + \frac{\partial \dot{x}^k}{\partial \dot{y}^i}\delta \dot{y}^i\right)\right) dt
\end{aligned}$$

Since x^i is a function of y^k and t only, $\frac{\partial x^k}{\partial \dot{y}^i} = 0$ and the second term in the first parentheses vanishes.

Now we need two identities. Explicitly expanding the velocity, $\dot{x}^k$, the chain rule gives:

$$\begin{aligned}
\dot{x}^k &= \frac{dx^k}{dt} \\
&= \frac{d}{dt} x^k\left(y^i(t), t\right) \\
&= \frac{\partial x^k}{\partial y^i}\dot{y}^i + \frac{\partial x^k}{\partial t} \qquad (1)
\end{aligned}$$

so differentiating, we have one identity,

$$\frac{\partial \dot{x}^k}{\partial \dot{y}^i} = \frac{\partial x^k}{\partial y^i}$$

For the second identity, we differentiate eq.(1) for the velocity with respect to y^i :

$$\begin{aligned}
\frac{\partial \dot{x}^k}{\partial y^i} &= \frac{\partial^2 x^k}{\partial y^i \partial y^j}\dot{y}^j + \frac{\partial^2 x^k}{\partial y^i \partial t} \\
&= \frac{\partial}{\partial y^j}\left(\frac{\partial x^k}{\partial y^i}\right)\dot{y}^j + \frac{\partial}{\partial t}\left(\frac{\partial x^k}{\partial y^i}\right) \\
&= \frac{d}{dt}\frac{\partial x^k}{\partial y^i}
\end{aligned}$$

Now return and substitute into the variation

$$\begin{aligned}
0 &= \delta S \\
&= \sum_{i,k=1}^{N} \int_C \left(\frac{\partial L}{\partial x^k}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{\partial \dot{x}^k}{\partial y^i}\delta y^i + \frac{\partial \dot{x}^k}{\partial \dot{y}^i}\delta \dot{y}^i\right)\right) dt \\
&= \sum_{i,k=1}^{N} \int_C \left(\frac{\partial L}{\partial x^k}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial L}{\partial \dot{x}^k}\left(\left(\frac{d}{dt}\frac{\partial x^k}{\partial y^i}\right)\delta y^i + \frac{\partial x^k}{\partial y^i}\delta \dot{y}^i\right)\right) dt \\
&= \sum_{i,k=1}^{N} \int_C \left(\frac{\partial L}{\partial x^k}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial L}{\partial \dot{x}^k}\frac{d}{dt}\left(\frac{\partial x^k}{\partial y^i}\delta y^i\right)\right) dt
\end{aligned}$$

Finally, integrate the final term by parts,

$$\begin{aligned}
\sum_{i,k=1}^{N}\int_C \frac{\partial L}{\partial \dot{x}^k}\frac{d}{dt}\left(\frac{\partial x^k}{\partial y^i}\delta y^i\right)dt &= \sum_{i,k=1}^{N}\int_C \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\frac{\partial x^k}{\partial y^i}\delta y^i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\frac{\partial x^k}{\partial y^i}\delta y^i\right)dt \\
&= \sum_{i,k=1}^{N}\left(\left(\frac{\partial L}{\partial \dot{x}^k}\frac{\partial x^k}{\partial y^i}\delta y^i\right)_{final} - \left(\frac{\partial L}{\partial \dot{x}^k}\frac{\partial x^k}{\partial y^i}\delta y^i\right)_{initial} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\frac{\partial x^k}{\partial y^i}\delta y^i\right)dt \\
&= \sum_{i,k=1}^{N}\left(-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\frac{\partial x^k}{\partial y^i}\delta y^i\right)dt
\end{aligned}$$

where δy_i vanishes at the endpoints. The vanishing variation now becomes

$$0 = \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}\delta y^i dt$$

The initial equality of the two forms of the action, $S\left[y^i\right] = S\left[x^i\left(y^k\right)\right]$ implies $\delta S\left[y^i\right] = \delta S\left[x^i\left(y^k\right)\right]$ and therefore

$$\begin{aligned}
\sum_{k=1}^{N}\int_C \left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k}\right)\right)\delta y^k dt - \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}\delta y^i dt &= 0 \\
\sum_{k=1}^{N}\int_C \left[\left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k}\right)\right) - \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}\right]\delta y^i dt &= 0
\end{aligned}$$

and the independence and arbitrariness of the variation, δy^i implies covariance:

$$\left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k}\right)\right) = \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}$$

The conclusion we reach is that no matter what coordinates q^i we choose for a problem, we may always write the equation of motion as

$$\frac{\partial L}{\partial q^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right) = 0$$

The same is true of the action. Rather than writing the Euler-Lagrange equation, we may write the action as the integral of the Lagrangian and write the Lagrangian in terms of whatever coordinates we choose,

$$S\left[q^i\right] = \int_{t_1}^{t_2} L\left(q^i, \dot{q}^i, t\right)dt$$