Constraints

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We are often interested in problems which do not allow all particles a full range of motion, but instead restrict motion to some subspace. When constrained motion can be described in this way, there is a simple technique for formulating the problem.

Subspaces of constraint may be described by relationships between the coordinates,

$$f\left(x^i, t\right) = 0$$

The trick is to introduce f into the problem in such a way that it must vanish in the solution. Our understanding of the Euler-Lagrange equation as the covariant form of Newton's second law tells us how to do this. Since the force that maintains the constraint must be orthogonal to the surface $f = 0$, it will be in the direction of the gradient of f and we can write

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \lambda \frac{\partial f}{\partial x^i}$$

where $\lambda = \lambda\left(x^i, \dot{x}^i, t\right)$ determines the amplitude of the gradient required to provide the constraining force. In addition, we need the constraint itself.

Remarkably, both the addition of $\lambda \frac{\partial f}{\partial x^i}$ and the constraint itself follow as a variation of a slightly altered form of the action. Since f itself is independent of the velocity, the simple replacement of the action by

$$S = \int \left(L + \lambda f\right) dt$$

means the the variation of S now gives

$$\delta S = \int \left(\left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{\partial L}{\partial x^i} + \lambda \frac{\partial f}{\partial x^i}\right)\delta x^i + f\delta\lambda\right) dt$$

where we *treat λ as an independent degree of freedom*. Thus, the variation $\delta\lambda$ is independent of the n coordinate variations δx^i and we get $n+1$ equations,

$$
\begin{aligned}
0 &= -\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{\partial L}{\partial x^i} + \lambda\frac{\partial f}{\partial x^i}\\
0 &= f
\end{aligned}
$$

These are exactly what we require – the extra equation gives just enough information to determine λ, while the addition to the Euler-Lagrange equation is the force of constraint.

Thus, by increasing the number of degrees of freedom of the problem by one for each constraint, we include the constraint while allowing free variation of the action. In exchange for the added equation of motion, we learn that the force required to maintain the constraint is

$$F^i_{constraint} = \lambda g^{ij}\frac{\partial f}{\partial x^i}$$

The advantage of treating constraints in this way is that we now may carry out the variation of the coordinates freely, as if all motions were possible. The variation of λ, called a *Lagrange multiplier*, brings in the constraint automatically. In the end, we will have the n Euler-Lagrange equations we started with (assuming an initial N degrees of freedom), plus an additional equation for each Lagrange multiplier.

When the constraint surface is fixed in space the constraint force never does any work since there is never any motion in the direction of the gradient of f. If there is time dependence of the surface then work will be done. Because f remains zero its total time derivative vanishes so

$$
\begin{aligned}
0 &= \frac{df}{dt} \\
&= \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial t}
\end{aligned}
$$

or multiplying by λdt and integrating,

$$
\begin{aligned}
W &= \int \mathbf{F}_{constraint} \cdot d\mathbf{x} \\
&= \sum_{i=1}^{n} \int \lambda \frac{\partial f}{\partial x^i} dx^i \\
&= -\int \lambda \frac{\partial f}{\partial t} dt
\end{aligned}
$$

Thus, the Lagrange multiplier allows us to compute the work done by a moving constraint surface.

Example As a simple example, consider the motion of a particle under the influence of gravity, $V = mgz$, constrained to a plane inclined at an angle θ. The angled surface satisfies $\tan\theta = \frac{z}{x}$, so

$$
f(x, z) = z - x\tan\theta = 0
$$

where θ is a fixed angle. We write the action as

$$
S = \int \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - mgz + \lambda\left(z - x\tan\theta\right) \right) dt
$$

Because y is cyclic we immediately have

$$
p_y = m\dot{y} = mv_{0y} = const.
$$

so that

$$
y = y_0 + v_{0y}t
$$

Because $\frac{\partial L}{\partial t} = 0$, we also have conservation of energy,

$$
\begin{aligned}
E &= \sum_i \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L \\
&= \frac{1}{2}m\dot{\mathbf{x}}^2 + mgz - \lambda\left(z - x\tan\theta\right)
\end{aligned}
$$

Varying x, z and λ we have three further equations,

$$
\begin{aligned}
0 &= m\ddot{x} + \lambda\tan\theta \\
0 &= m\ddot{z} + mg - \lambda \\
0 &= z - x\tan\theta
\end{aligned}
$$

Combining the constraint with the conservation of energy, the conservation law reduces to

$$E = \frac{1}{2}m\dot{\mathbf{x}}^2 + mgz - \lambda(z - x\tan\theta)$$
$$= \frac{1}{2}m\dot{\mathbf{x}}^2 + mgz$$

This shows that for this example the constraint contributes no energy.

To solve the x and z equations, we must eliminate λ. Differentiate the constraint equation twice to get

$$\ddot{z} - \ddot{x}\tan\theta = 0$$

Substituting for $\ddot{x}$ and $\ddot{z}$ from the first two equations of motion, this becomes

$$0 = \ddot{z} - \ddot{x}\tan\theta$$
$$= -\frac{1}{m}(mg - \lambda) - \left(-\frac{\lambda}{m}\tan\theta\right)\tan\theta$$
$$= -g + \frac{\lambda}{m} + \frac{\lambda}{m}\tan^2\theta$$
$$= -g + \frac{\lambda}{m\cos^2\theta}$$

giving the Lagrange multiplier

$$\lambda = mg\cos^2\theta$$

In this case, λ is constant.

Eliminating λ from the x and z equations gives

$$0 = \ddot{x} + g\cos\theta\sin\theta$$
$$0 = \ddot{z} + g\sin^2\theta$$

and these are immediately integrated to give

$$x = x_0 + v_{0x}t + \frac{1}{2}\cos\theta\sin\theta gt^2$$
$$z = z_0 + v_{0z}t - \frac{1}{2}\sin^2\theta gt^2$$

The constraint force is given by

$$F^i = \lambda g^{ij}\frac{\partial f}{\partial x^j}$$
$$= mg\cos^2\theta(-\tan\theta, 0, 1)$$
$$= mg\cos\theta(-\sin\theta, 0, \cos\theta)$$

Notice that the magnitude is the normal force, $mg\cos\theta$ as expected, with the unit vector $\hat{\mathbf{n}} = (-\sin\theta, 0, \cos\theta)$ being the normal to the plane.