Central Forces I: Simplifying

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1 Center of mass coordinates

We consider the general 2-body problem, where the force between two point particles of masses M and m is derivable from a central potential, $V(r)$, where r is the distance between the two masses. We begin with the Cartesian form of the action,

$$S = \int_{t_1=0}^{t_2=t} \frac{1}{2}M\dot{\mathbf{X}}^2 + \frac{1}{2}m\dot{\mathbf{x}}^2 - V\left(\sqrt{(\mathbf{X}-\mathbf{x})^2}\right)$$

which is dependent upon six coordinates. We make a simplification by introducing new coordinates, the position $\mathbf{R} \equiv \frac{M\mathbf{X}+m\mathbf{x}}{M+m}$ of the center of mass, and the separation vector, $\mathbf{r} \equiv \mathbf{x} - \mathbf{X}$ from M to m. In terms of these we may solve for $\mathbf{X}$ and $\mathbf{x}$:

$$\begin{aligned}
(M+m)\,\mathbf{R} &= M\mathbf{X} + m\mathbf{x} \\
&= M\mathbf{X} + m\,(\mathbf{r}+\mathbf{X}) \\
(M+m)\,\mathbf{R} - m\mathbf{r} &= (M+m)\,\mathbf{X} \\
\mathbf{X} &= \mathbf{R} - \frac{m\mathbf{r}}{M+m}
\end{aligned}$$

and then

$$\begin{aligned}
\mathbf{x} &= \mathbf{r} + \mathbf{X} \\
&= \mathbf{r} + \mathbf{R} - \frac{m\mathbf{r}}{M+m} \\
&= \mathbf{R} + \frac{M}{M+m}\mathbf{r}
\end{aligned}$$

In terms of these, the velocities become

$$\begin{aligned}
\dot{\mathbf{X}} &= \dot{\mathbf{R}} - \frac{m\dot{\mathbf{r}}}{M+m} \\
\dot{\mathbf{x}} &= \dot{\mathbf{R}} + \frac{M\dot{\mathbf{r}}}{M+m}
\end{aligned}$$

so the kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}M\dot{\mathbf{X}}^2 + \frac{1}{2}m\dot{\mathbf{x}}^2 \\
&= \frac{1}{2}M\left(\dot{\mathbf{R}}^2 - \frac{2m\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}}{M+m} + \frac{m^2\dot{\mathbf{r}}^2}{(M+m)^2}\right) + \frac{1}{2}m\left(\dot{\mathbf{R}}^2 + \frac{2M\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}}{M+m} + \frac{M^2\dot{\mathbf{r}}^2}{(M+m)^2}\right) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 - \frac{Mm}{M+m}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}} + \frac{1}{2}\frac{Mm^2\dot{\mathbf{r}}^2}{(M+m)^2} + \frac{1}{2}m\dot{\mathbf{R}}^2 + \frac{mM\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}}{M+m} + \frac{1}{2}\frac{mM^2\dot{\mathbf{r}}^2}{(M+m)^2} \\
&= \frac{1}{2}(M+m)\,\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{mM}{M+m}\dot{\mathbf{r}}^2
\end{aligned}$$

We define the *reduced mass* and the *total mass*

$$\begin{aligned} \mu &\equiv \frac{mM}{M+m} \\ M_0 &\equiv M+m \end{aligned}$$

and the action becomes

$$\begin{aligned} S &= \int_0^t \left(\frac{1}{2} M_0 \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r) \right) dt \\ &= \int_0^t \frac{1}{2} M_0 \dot{\mathbf{R}}^2 dt + \int_0^t \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r) \right) dt \end{aligned}$$

The action has separated into two decoupled terms, S_R and S_r. We may write the velocities in either Cartesian,

$$\begin{aligned} \dot{\mathbf{R}}^2 &= \dot{R}_x^2 + \dot{R}_y^2 + \dot{R}_z^2 \\ \dot{\mathbf{r}}^2 &= \dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2 \end{aligned}$$

or spherical,

$$\begin{aligned} \dot{\mathbf{R}}^2 &= \dot{R}^2 + R^2 \dot{\Theta}^2 + R^2 \sin^2 \Theta \dot{\Phi}^2 \\ \dot{\mathbf{r}}^2 &= \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2 \end{aligned}$$

2 Conserved quantities

We consider consequences of Noether's theorem S.

First, we notice that all components of the center of mass are cyclic, $\frac{\partial L}{\partial R^i} = 0$, so that

$$\mathbf{P} = M_0 \dot{\mathbf{R}}$$

is conserved. Integrating, the motion of the center of mass proceeds at constant velocity,

$$\mathbf{R} = \mathbf{R}_0 + \frac{\mathbf{P}}{M_0} t$$

This eliminates three of the six degrees of freedom.

Now we are left with

$$S_r = \int_0^t \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r) \right) dt$$

and we consider a rotational variation,

$$\delta_\varepsilon r^i = \varepsilon^i{}_j r^j$$

where $\varepsilon_{ij} = -\varepsilon_{ji}$. We find

$$\delta_\varepsilon S_r = \int_0^t \left(\mu \dot{\mathbf{r}} \cdot \delta_\varepsilon \dot{\mathbf{r}} - \frac{\partial V}{\partial r} \delta_\varepsilon r \right) dt$$

Working out the variations, we have

$$\begin{aligned}
\delta_\varepsilon r &= \delta_\varepsilon \sqrt{r_x^2 + r_y^2 + r_z^2} \\
&= \frac{1}{\sqrt{r_z^2 + r_y^2 + r_z^2}} \left(r_x \delta_\varepsilon r_x + r_y \delta_\varepsilon r_y + r_z \delta_\varepsilon r_z \right) \\
&= \frac{1}{\sqrt{r_z^2 + r_y^2 + r_z^2}} r^i \varepsilon_{ij} r^j \\
&= 0
\end{aligned}$$

since $\varepsilon_{ij} r^i r^j = 0$. For the kinetic term, we similarly have

$$\dot{\mathbf{r}} \cdot \delta_\varepsilon \dot{\mathbf{r}} = \dot{r}^i \varepsilon_{ij} \dot{r}^j = 0$$

and the action is rotationally invariant.

Now we use Noether's theorem to conclude that

$$\frac{\partial L}{\partial \dot{r}^i} \varepsilon^i{}_j r^j$$

is conserved. Since athe arbitrary antisymmetric matrix, ε_{ij} may be written as

$$\varepsilon_{ij} = \varepsilon_{ijk} a^k$$

where a^k is an arbitrary constant vector, we have three conserved quantities,

$$L_k a^k \equiv -\mu \varepsilon_{ijk} \dot{r}^i r^j a^k$$

Since a^k is arbitrary and constant, we may identify the angular momentum, $\mathbf{L}$, as the cross product

$$\mathbf{L} = \mathbf{r} \times \mu \dot{\mathbf{r}}$$

Finally, L contains no explicit time dependence, so we have conserved energy.

3 The equation of motion

We use two of the conserved angular momenta immediately. The constancy of $\mathbf{L}$ means that the position $\mathbf{r}$ and reduced momentum $\mu\dot{\mathbf{r}}$ always lie in the same plane. To see this, choose initial coordinates so that both lie in the xy-plane. Then since $\mathbf{L} = L\mathbf{k}$ from the initial conditions, we have

$$\begin{aligned}
0 &= \mathbf{k} \times \mathbf{L} \\
&= \mathbf{k} \times (\mathbf{r} \times \mu \dot{\mathbf{r}}) \\
&= \mathbf{r} (\mathbf{k} \cdot \mu \dot{\mathbf{r}}) - \mu \dot{\mathbf{r}} (\mathbf{k} \cdot \mathbf{r})
\end{aligned}$$

Since $\mathbf{r}$ and $\dot{\mathbf{r}}$ must lie in distinct directions (unless $\mathbf{L} = 0$, in which case they are always along a single line), we must have both $\mathbf{k} \cdot \mu\dot{\mathbf{r}} = 0$ and $\mathbf{k} \cdot \mathbf{r} = 0$ at all times.

Given the planar character of the motion, we choose the $\theta = \frac{\pi}{2}$ plane for the initial directions, and this angle cannot change so $\dot{\theta} = 0, \sin\theta = 0$. Writing the action in these spherical coordinates then gives

$$S_r = \int_0^t \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r) \right) dt$$

$$
\begin{aligned}
&= \int_0^t \left(\frac{1}{2}\mu \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\varphi}^2 \right) - V(r) \right) dt \\
&= \int_0^t \left(\frac{1}{2}\mu \left(\dot{r}^2 + r^2\dot{\varphi}^2 \right) - V(r) \right) dt
\end{aligned}
$$

We are left with only two coordinates, with φ cyclic. The conserved angluar momentum (the remaining degree of freedom of $\mathbf{L}$) is

$$L = \mu r^2 \dot{\varphi}$$

and the sole equation of motion from varying r is

$$-\mu\ddot{r} + \mu r\dot{\varphi}^2 - \frac{\partial V}{\partial r} = 0$$

Substituting $\dot{\varphi} = \frac{L}{\mu r^2}$ we have a single, ordinary differential equation,

$$\mu\ddot{r} - \frac{L^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0$$