Central Forces

October 8, 2012

By moving to the center of mass system, $\mathbf{R} = 0$, the general two-body problem with force $f(r) = -\frac{dV}{dr}$ reduces to a 2-dimensional problem in a plane with action

$$S = \int \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. The equations of motion are then

$$\begin{aligned} \mu\ddot{r} - \mu r\dot{\varphi}^2 - f(r) &= 0 \\ \frac{d}{dt}\left(\mu r^2\dot{\varphi}\right) &= 0 \end{aligned}$$

We may integrate these directly, or use our the symmetries of the system to write two constants of the motion direrctly. Since there is no explicit time dependence in the lagrangian, energy is conserved,

$$\begin{aligned} E &= \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \\ &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r) \end{aligned}$$

and since φ is cyclic, the angular momentun is conserved,

$$L_\varphi = \mu r^2\dot{\varphi}$$

Notice that the total angular momentum $\mathbf{L} = L_\varphi\hat{\mathbf{n}}$ actually provides three constants of motion. Two of these fix the direction of the unit vector $\hat{\mathbf{n}}$, and therefore the oriention of the plane of motion, while the third is the magnitude, L_φ.

We have the immediate solution for $\dot{\varphi}$,

$$\dot{\varphi} = \frac{L_\varphi}{\mu r^2}$$

and this allows us to write the energy equation entirely in terms of r and $\dot{r}$,

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L_\varphi^2}{2\mu r^2} + V(r)$$

Solving for $\frac{dr}{dt}$,

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - \frac{L_\varphi^2}{2\mu r^2} - V(r)\right)}$$

and integrating, we have a formal solution,

$$\begin{aligned} t &= \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{L_\varphi^2}{2\mu r^2} - V(r)\right)}} \\ \varphi &= \varphi_0 + \int_0^t \frac{L_\varphi}{\mu r^2(t)}dt \end{aligned}$$

where the first equation determines $r(t)$, which is then used in the second.

We may also combine the orbital equations to get the shape of the orbit,

$$\begin{aligned}
\frac{dr}{d\varphi} &= \frac{\dot{r}}{\dot{\varphi}} = \frac{\mu r^2}{L_\varphi}\sqrt{\frac{2}{\mu}\left(E - \frac{L_\varphi^2}{2\mu r^2} - V(r)\right)} \\
\varphi &= \int \frac{L_\varphi dr}{\mu r^2 \sqrt{\frac{2}{\mu}\left(E - \frac{L_\varphi^2}{2\mu r^2} - V(r)\right)}}
\end{aligned}$$

Power law potentials

Consider the class of attractive potentials

$$V(r) = \begin{cases} Ar^n & +powers \\ -\frac{A}{r^n} & -powers \end{cases}$$

for any positive or negative integer n. We can understand qualitative properties of the solutions from these expressions. From the velocity, we see there may be one or two turning points, i.e., points where $\frac{dr}{dt} = 0$. These are given by

$$\begin{aligned}
\sqrt{\frac{2}{\mu}\left(E - \frac{L_\varphi^2}{2\mu r^2} - V(r)\right)} &= 0 \\
2\mu r^2 E - L_\varphi^2 - 2\mu A r^{n+2} &= 0
\end{aligned}$$

This is a polynomial equation of order $s = max(2, n+2)$. Regarded in the complex plane, such an equation has exactly s solutions, all lying equally spaced on a circle about the origin in the complex plane. Therefore, there can be at most 2 real solutions. For positive powers, $n > 0$, the expression is negative as $r \longrightarrow 0$, but must eventually become positive as $r \longrightarrow \infty$. Therefore, there is at least one zero. When the power is negative, $n < 0$, the same is true, with the expression tending to $-\infty$ as $r \longrightarrow 0$. We conclude that there are two types of motion, depending on whether there is one solution or there are two.

For repulsive potentials, the motion is away from the center of force so there will be either one turning point or none, depending on the initial velocity.

If there are two turning points, the motion is bounded between them and we have *orbits*. If there is one turning point, we have *scattering*.

We may also look at the motion as that of a free particle in an effective potential

$$V_{eff} = \frac{L_\varphi^2}{2\mu r^2} + V(r)$$

For $n > 0$, this potential has extrema at

$$\begin{aligned}
0 &= -\frac{L_\varphi^2}{\mu r^3} + nAr^{n-1} \\
\frac{L_\varphi^2}{nA\mu} &= r^{n+2}
\end{aligned}$$

so there is a single minimum, since we always have $r > 0$. For $n < 0$,

$$\begin{aligned}
0 &= -\frac{L_\varphi^2}{\mu r^3} + \frac{nA}{r^{n+1}} \\
r^{n-2} &= \frac{n\mu A}{L_\varphi^2}
\end{aligned}$$

and we again have a single minimum. Therefore, for energy chosen at this minimum, we may always have circular orbits. We now examine when perturbations of these circular orbits are closed.

Newtonian gravity

Let the force be given by Newton's law of universal gravitation,

$$\mathbf{F} = -\frac{G\mathrm{Mm}}{r^2}\hat{\mathbf{r}}$$

Then $V = -\frac{GMm}{r}$ and for general initial conditions, we have

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{L_\varphi dr}{\mu r^2 \sqrt{\frac{2}{\mu}\left(E - \frac{L_\varphi^2}{2\mu r^2} + \frac{GMm}{r}\right)}}$$

Choose the initial conditions so that $r(0) = r_0 = r_{min}$ when $\varphi(0) = \varphi_0 = 0$. Integrating, we substitute $\frac{1}{u} = r$,

$$\begin{aligned}
\varphi &= -\int_{r_0}^{r} \frac{L_\varphi du}{\mu\sqrt{\frac{2}{\mu}\left(E - \frac{L_\varphi^2}{2\mu}u^2 + GMmu\right)}} \\
\varphi &= -\int_{r_0}^{r} \frac{du}{\sqrt{-\left(u^2 - \frac{2\mu}{L_\varphi^2}GMmu - \frac{2\mu E}{L_\varphi^2}\right)}}
\end{aligned}$$

Notice that the number of turning points in the orbit depends on the number of zeros of the denominator, given by

$$\begin{aligned}
0 &= u^2 - \frac{2\mu}{L_\varphi^2}GMmu - \frac{2\mu E}{L_\varphi^2} \\
u &= \frac{\mu}{L_\varphi^2}GMm \pm \sqrt{\frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2}}
\end{aligned}$$

For negative total energy, three will be two roots as long as the energy is in the range

$$-\frac{\mu}{2L_\varphi^2}G^2M^2m^2 \le E < 0$$

and u is in the range

$$\frac{\mu}{L_\varphi^2}GMm - \sqrt{\frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2}} \le u \le \frac{\mu}{L_\varphi^2}GMm + \sqrt{\frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2}}$$

In this case we have bound orbits. For positive energy, only the upper sign gives an allowed root, since u must be positive, and we have scattering,

$$0 \le u \le \frac{\mu}{L_\varphi^2}GMm + \sqrt{\frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2}}$$

Complete the square,

$$u^2 - \frac{2\mu}{L_\varphi^2}GMmu - \frac{2\mu E}{L_\varphi^2} = \left(u - \frac{GMm\mu}{L_\varphi^2}\right)^2 - \left(\frac{GMm\mu}{L_\varphi^2}\right)^2 - \frac{2\mu E}{L_\varphi^2}$$

Define

$$
\begin{aligned}
a &= \frac{GMm\mu}{L_\varphi^2} \\
b^2 &= \left(\frac{GMm\mu}{L_\varphi^2}\right)^2 + \frac{2\mu E}{L_\varphi^2} \\
&> 0
\end{aligned}
$$

where the positive definiteness of b^2 is the condition for bound orbits found above. Then

$$
\varphi = -\int_{r_0}^{r} \frac{du}{\sqrt{b^2 - (u-a)^2}}
$$

Setting $\xi = u - a$,

$$
\varphi = -\int_{r_0}^{r} \frac{du}{\sqrt{b^2 - \xi^2}}
$$

Now, check the range of ξ. From the range for u for bound orbits, we have

$$
\begin{aligned}
-\sqrt{\frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2}} &\le u - \frac{\mu}{L_\varphi^2}GMm \le +\sqrt{\frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2}} \\
\xi^2 = (u-a)^2 &\le \frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2} \\
\xi^2 &\le b^2
\end{aligned}
$$

with the same result for scattering,

$$
\begin{aligned}
0 \le u &\le \frac{\mu}{L_\varphi^2}GMm + \sqrt{\frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2}} \\
(u-a)^2 &\le \frac{\mu^2}{L_\varphi^4}G^2M^2m^2 + \frac{2\mu E}{L_\varphi^2} \\
\xi^2 &\le b^2
\end{aligned}
$$

and we see that we may always use a trigonometric substitution, $\xi = b\sin\theta$. This allows us to complete the integral,

$$
\begin{aligned}
\varphi &= -\int_{r_0}^{r} \frac{b\cos\theta d\theta}{\sqrt{b^2 - b^2\sin^2\theta}} \\
&= -\theta - \theta_0 \\
&= -\sin^{-1}\frac{\xi}{b} - \sin^{-1}\frac{\xi_0}{b} \\
&= -\sin^{-1}\frac{u-a}{b} - \sin^{-1}\frac{u_0-a}{b} \\
&= -\sin^{-1}\frac{\frac{1}{r}-a}{b} - \sin^{-1}\frac{\frac{1}{r_0}-a}{b} \\
&= -\sin^{-1}\frac{1-ar}{br} - \sin^{-1}\frac{1-ar_0}{br_0}
\end{aligned}
$$

Let $\lambda = \sin^{-1} \frac{1-ar_0}{br_0}$. Then

$$\begin{aligned} \sin(\varphi + \lambda) &= \frac{ar-1}{br} \\ 1 &= r\left(a - b\sin(\varphi+\lambda)\right) \end{aligned}$$

and finally

$$r = \frac{1}{a - b\sin(\varphi+\lambda)}$$

Since our initial condition is that $r = r_{min}$ when $\varphi = 0$, and because the expression on the right is minimal when $\sin(\varphi+\lambda) = -1$, we set $\varphi = 0$ and require $\sin\lambda = -1$. Therefore, $\lambda = -\frac{\pi}{2}$ and we have

$$r = \frac{1}{a + b\cos\varphi}$$

This equation is often written in terms of the eccentricity, ε, and the semi-major axis, A,

$$r = \frac{A\left(1-e^2\right)}{1 + e\cos\varphi}$$

so we have

$$\begin{aligned} a &= \frac{1}{A\left(1-e^2\right)} \\ b &= \frac{e}{A\left(1-e^2\right)} \end{aligned}$$

Inverting and substituting the constants of motion,

$$\begin{aligned} e &= \frac{b}{a} \\ &= \frac{\sqrt{\left(\frac{GMm\mu}{L_\varphi^2}\right)^2 + \frac{2\mu E}{L_\varphi^2}}}{\frac{GMm\mu}{L_\varphi^2}} \\ &= \sqrt{1 + 2\mu E\left(\frac{L_\varphi}{GMm\mu}\right)^2} \\ &= \sqrt{1 + \frac{2EL_\varphi^2}{(GMm)^2\mu}} \\ A &= \frac{1}{a\left(1-e^2\right)} \\ &= \frac{a}{a^2 - b^2} \\ &= \frac{\frac{GMm\mu}{L_\varphi^2}}{\left(\frac{GMm\mu}{L_\varphi^2}\right)^2 - \left(\frac{GMm\mu}{L_\varphi^2}\right)^2 - \frac{2\mu E}{L_\varphi^2}} \\ &= -\frac{GMm}{2E} \end{aligned}$$

Bound orbits

For bound orbits, the total energy, E, is negative, so

$$\begin{aligned} e &= \sqrt{1 - \frac{2\,|E|\,L_\varphi^2}{(GMm)^2\,\mu}} \\ A &= \frac{GMm}{2\,|E|} \end{aligned}$$

Recall our observation that for bound orbits the energy must be in the range

$$-\frac{\mu}{2L_\varphi^2} G^2 M^2 m^2 \le E < 0$$

We now see that this corresponds exactly to eccentricity in the range

$$0 \le e < 1$$

with the minimum energy giving a circular orbit. The equation for the orbit describes an ellipse, with $r = 0$ lying at one focus of the ellipse.

Scattering

For scattering, we have positive energy, so the eccentricity is greater than 1, and A is negative,

$$\begin{aligned} e &= \sqrt{1 + \frac{2EL_\varphi^2}{(GMm)^2\,\mu}} \\ A &= -\frac{GMm}{2E} \end{aligned}$$

The solution for $r(\varphi)$ may now be written as

$$\begin{aligned} r &= \frac{A\left(1-e^2\right)}{1+e\cos\varphi} \\ &= \frac{(-A)\left(e^2-1\right)}{1+e\cos\varphi} \\ &= \frac{|A|\left(e^2-1\right)}{1+e\cos\varphi} \end{aligned}$$

Notice that this expression diverges for certain values of φ,

$$\begin{aligned} 1 + e\cos\varphi &= 0 \\ \cos\varphi &= -\frac{1}{e} \end{aligned}$$

This restricts φ to the range

$$\frac{\pi}{2} + \cos^{-1}\left(\frac{1}{e}\right) \le \varphi \le \frac{3\pi}{2} - \cos^{-1}\left(\frac{1}{e}\right)$$

and r reaches infinity at the limits. The shape is now hyperbolic; r comes in toward the scattering center from a large distance, swings around and flies off in a new direction, escaping to arbitrarily large distance.

It may be shown that the limiting, marginally unbound case is a parabola. We therefore have the possibility of circles, parabolas, ellipses and hyperbolas – exactly the *conic sections*. The conic sections are the intersections of a plane with a cone.