The Brachistochrone I: Roller coaster

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Consider the first great hill and dip of a roller coaster. We would like to optimize its shape, from the beginning of the drop at point A to the top of the next hill at B in such a way that the time required for the trip from A to B is a minimum. The only force that does any work is the conservative gravitational force, $-mg\hat{\mathbf{k}}$, but in addition there are unknown forces of constraint holding the car to the track.

1 The action and conserved quantities

Let a car of mass m move along the tack without friction. Choose the z coordinate positive downward so the coordinates of the initial and final points are

$$
\begin{aligned}
\mathbf{x}_A &= (0, 0, h_A) \\
\mathbf{x}_B &= (L, 0, h_B)
\end{aligned}
$$

with $h_A > h_B$. It is clear that the motion stays in the xz-plane because the potential,

$$V = -mgz$$

does not depend on y. Therefore, movement in the y-direction cannot make the car go faster, but always adds distance, thereby increasing the time. The kinetic energy is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right)$ so the action is

$$S = \int\limits_{0}^{t_f} \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) + mgz\right] dt$$

Since x is cyclic, we have

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

constant. Clearly this is not the case for the constrained motion, however. Though it starts with zero velocity, the car ultimately moves to the right, the constraint forces having accelerated it in that direction.

On the other hand, the forces of constraint do no work, and $\frac{\partial L}{\partial t} = 0$, so we expect energy to be conserved,

$$
\begin{aligned}
E &= \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{z}}\dot{z} - L \\
&= \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) - mgz
\end{aligned}
$$

We may choose the energy to be measured from the initial point, $z = 0, v = 0$ so that the constant is $E = 0$. Then the total velocity, $v = \sqrt{\dot{x}^2 + \dot{z}^2}$, is therefore given by

$$v = \sqrt{2gz}$$

However, we do not have either the path or the force of constraint, so we can go no further with the Lagrangian.

2 The time functional

Instead, we write the time directly. Integrating $v = \frac{ds}{dt}$ where ds is an infinitesimal section of the track, we have

$$t_f = \int_0^{s_f} \frac{ds}{v}$$

where we may write ds as

$$ds = \sqrt{dx^2 + dz^2}$$

If we parameterize the path by

$$C(\lambda) = (x(\lambda), z(\lambda))$$

then we have

$$
\begin{aligned}
t_f &= \int_0^{s_f} \frac{ds}{v} \\
&= \int_0^{s_f} \frac{\sqrt{dx^2 + dz^2}}{\sqrt{2gz}} \\
&= \int_0^{s_f} \frac{\sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2}}{\sqrt{2gz}} \, d\lambda
\end{aligned}
$$

Thus, t_f is a functional of $x(\lambda)$ and $z(\lambda)$. We may choose λ to be any parameter which is monotonic along the path. In this case it is simplest to let $\lambda = x$, thinking of $z = z(x)$ as the curve. Then

$$
t_f = \int_0^{s_f} \frac{\sqrt{1 + \left(\frac{dz}{dx}\right)^2}}{\sqrt{2gz}} \, dx
$$

Let the dot denote differentiation with respect to x, i.e., $\dot{z} = \frac{dz}{dt}$. Then

$$
t_f = \int_0^{s_f} \sqrt{\frac{1 + \dot{z}^2}{2gz}} \, dx
$$

3 Variation

Since the integrand is a function of z and $\dot{z}$, the extremum is given by the Euler-Lagrange equation,

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{\partial}{\partial \dot{z}} \sqrt{\frac{1 + \dot{z}^2}{2gz}}\right) - \frac{\partial}{\partial z}\sqrt{\frac{1 + \dot{z}^2}{2gz}} &= 0 \\
\frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right) + \frac{1}{2}\sqrt{1 + \dot{z}^2}\frac{1}{\sqrt{2g}z^{3/2}} &= 0 \\
\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right) + \frac{1}{2}\sqrt{1 + \dot{z}^2}\frac{1}{\sqrt{2g}z^{3/2}}\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}} &= 0 \\
\left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right)\frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right) + \frac{\dot{z}}{4gz^2} &= 0
\end{aligned}
$$

Then

$$\int\left(\frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}}\right)d\left(\frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}}\right) = -\int\frac{dz}{4gz^2}$$

$$\frac{1}{2}\left(\frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}}\right)^2 = \frac{1}{4gz} - C$$

$$\frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}} = \sqrt{\frac{1}{2gz} - 2C}$$

$$\frac{\dot{z}}{\sqrt{1+\dot{z}^2}} = \sqrt{1-4Cgz}$$

$$\frac{\dot{z}^2}{1+\dot{z}^2} = 1-4Cgz$$

$$\dot{z}^2 = (1-4Cgz)\left(1+\dot{z}^2\right)$$

$$4Cgz\dot{z}^2 = 1-4Cgz$$

$$\dot{z}^2 = \frac{1-4Cgz}{4Cgz}$$

$$\frac{dz}{dx} = \sqrt{\frac{1-4Cgz}{4Cgz}}$$

So finally

$$\frac{dz}{dx} = \sqrt{\frac{1-az}{az}}$$

where $a = 4Cg$. This is straightforward to integrate, but we can show directly that it describes a cycloid, which may be parametrically written as

$$x = c\left(\phi - \sin\phi\right)$$
$$z = c\left(1 - \cos\phi\right)$$

Computing the differentials,

$$dx = c\left(1 - \cos\phi\right)d\phi$$
$$dz = c\sin\phi d\phi$$

Rewriting the right side in terms of z',

$$dx = zd\phi$$
$$dz = c\sqrt{1-\cos^2\phi}d\phi$$
$$= c\sqrt{1-\left(\frac{c-z}{c}\right)^2}d\phi$$
$$= \sqrt{2cz-z^2}d\phi$$

and therefore

$$\frac{dz}{dx} = \frac{\sqrt{2cz-z^2}}{z}$$
$$= \sqrt{\frac{1-\frac{1}{2c}z}{\frac{1}{2c}z}}$$

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so we have the same expression if we choose $2c = \frac{1}{a}$. The solution curve is therefore

$$x = \frac{1}{2a}(\phi - \sin\phi)$$

$$z = \frac{1}{2a}(1 - \cos\phi)$$

We cannot directly write $z(x)$ since the expression for $x(\phi)$ is transcendental. We can write $x(z)$ however, since

$$\cos\phi = 1 - 2az$$

$$\phi = \cos^{-1}(1 - 2az)$$

and therefore,

$$x = \frac{1}{2a}\left(\cos^{-1}(1 - 2az) - \sqrt{1 - (1 - 2az)^2}\right)$$

$$= \frac{1}{2a}\left(\cos^{-1}(1 - 2az) - 2\sqrt{az - a^2z^2}\right)$$

The parametric equation is easier to understand.