

Beyond the Second Law

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1 Introduction

There are several reasons for considering alternative formulations of Newton's second law:

1. General coordinate invariance
2. Symmetry and the resulting conservation laws
3. Constraints
4. There exist problems which resist solution using the second law
5. Connection with quantum mechanics

We are particularly interested in the Lagrangian and Hamiltonian formulations of mechanics, in which systematic approaches to these properties are realized. Briefly, the Lagrangian formulation arrives at the equations of motion by finding extrema of the *action*, $S(\mathbf{x}, t)$, given by

$$S(\mathbf{x}, t) = \int_{t_1}^{t_2} L(\mathbf{x}, t) dt$$

where $L(\mathbf{x}, t)$ is the *difference* between the kinetic and potential energies,

$$L(\mathbf{x}, t) = T - V$$

We discuss the four points above in turn below, then in the next set of Notes, introduce the Action and begin development of Lagrangian mechanics.

2 General coordinate invariance

You are already familiar with using different coordinate systems to describe physical problems. You have used Cartesian, cylindrical and spherical coordinates for problems with those symmetries. We have seen, in the example of the 2-dimensional harmonic oscillator that one coordinate system may make a problem much simpler than another.

The number of possible coordinate systems is unlimited. As long as we can locate any point in space unambiguously, we may use any triple of numbers. Changing coordinates gives us great power in solving problems. For example, consider possible coordinates in the plane. For the region above the diagonal line $y = x$ and to the right of the positive y -axis, let r be the radius of any circle, and consider the set of hyperbolas given by

$$y^2 - x^2 = \lambda^2$$

Each point in the given region has exactly one circle and one hyperbola passing through it. Therefore, specifying the pair (r, λ) uniquely determines any point in the region. To see this explicitly, we can find the (x, y) coordinates of the same point. Since $y > 0$, we have

$$y = +\sqrt{x^2 + \lambda^2}$$

We also know that

$$\begin{aligned} r &= +\sqrt{x^2 + y^2} \\ &= +\sqrt{2x^2 + \lambda^2} \end{aligned}$$

so that

$$2x^2 = r^2 - \lambda^2$$

and this is positive for any point in the region so we have

$$\begin{aligned} x &= +\sqrt{\frac{1}{2}(r^2 - \lambda^2)} \\ y &= +\sqrt{\frac{1}{2}(r^2 - \lambda^2) + \lambda^2} \\ &= +\sqrt{\frac{1}{2}(r^2 + \lambda^2)} \end{aligned}$$

and these values are unique throughout the region.

As a second example, consider the sinusoidal curves

$$y = \sin x + \alpha$$

together with the vertical lines

$$(x, y) = (\beta, y)$$

for all α, β . It is easy to see that each pair (α, β) determines exactly one point,

$$(x, y) = (\beta, \alpha + \sin \beta)$$

There are two important things to realize here:

1. Any set of parameters in continuous, one-to-one correspondence with points in space may serve as coordinates.
2. Our physical predictions must be independent of our choice of coordinates.

It follows from these points that we may choose coordinates that make our problem simpler to solve.

However, the second law changes if we change coordinates. If we replace x by a new coordinate y , with the relation between them given by a function $x(y)$, then in the y -coordinate Newton's second law in 1-dimension becomes

$$\begin{aligned} F &= m \frac{d^2 x}{dt^2} \\ &= m \frac{d}{dt} \left(\frac{dx}{dy} \dot{y} \right) \\ &= m \left(\frac{d^2 x}{dy^2} \dot{y}^2 + \frac{dx}{dy} \ddot{y} \right) \end{aligned}$$

which, in addition to the function $\frac{dx}{dy}$ multiplying the acceleration term, $m\ddot{y}$, has an additional term proportional to the velocity squared. Newton's second law is not coordinate invariant. As we shall see, it is possible to rewrite the second law in a form which does *not* change when we change coordinates.

3 Symmetries and the resulting conservation laws

Consider a single free particle in 1-dimension. It's equation of motion is

$$\frac{dp}{dt} = 0$$

so it's momentum is conserved, $p = mv = \text{const.}$ Now notice that we may write the equation of motion as

$$m \frac{d^2x}{dt^2} = 0$$

If we substitute $y = x + a$ for any constant a , we have the identical equation for y ,

$$m \frac{d^2y}{dt^2} = 0$$

This property is called *translational invariance*, and it is a *symmetry* of the system.

It turns out that this sort of relationship between symmetries and conservation laws is general. Noether's theorem establishes this using the Lagrangian formulation of mechanics, and the details give us a way to quickly identify symmetries and conserved quantities.

4 Constraints

Many problems we deal with involve fewer real degrees of freedom than three coordinates for each particle. For example, Atwood's machine, with two masses connected by a rope which passes over a pulley, has only one degree of freedom since the position of one mass determines the position of the other. Similarly, a block sliding down a plane does not leap from the plane, but is constrained to slide along the sloped surface. Yet in its present form, Newton's second law does not give us a systematic way to include such constraints. We are required to spot them one by one and insert them in any way we can.

In the Lagrangian formulation of mechanics, there is a standard technique which lets us handle many constraints. Specifically, the method of *Lagrange multipliers* works any time the constraint can be written by specifying one coordinate in terms of the others. Such constraints are called *holonomic*.

An example of a holonomic constraint is a particle of mass m moving in 2-dimensions, constrained to move on the surface of the hyperbola

$$y^2 - x^2 = c_0^2$$

under a force of gravity, $-mg\hat{\mathbf{j}}$. We can find adapted coordinates to the hyperbola of this problem. Let one coordinate be a series of nested hyperbolas including the given surface,

$$c^2 = y^2 - x^2$$

and the other be arc-length along each hyperbola. To find length along the hyperbola, notice that the differential of the equation for the hyperbola gives

$$\begin{aligned} 2ydy - 2xdx &= 0 \\ \frac{dy}{dx} &= \frac{x}{y} \end{aligned}$$

while Pythagorean arc-length is

$$\begin{aligned} du &= \sqrt{dx^2 + dy^2} \\ u(x, y) &= \int \sqrt{dx^2 + dy^2} \end{aligned}$$

$$\begin{aligned}
&= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int \sqrt{1 + \left(\frac{x}{y}\right)^2} dx \\
&= \int^x \sqrt{1 + \left(\frac{x^2}{c^2 + x^2}\right)} dx
\end{aligned}$$

The answer turns out to be an elliptic integral, but the important thing to note here is that we can now specify the constraint on the motion by fixing one of the coordinates,

$$v = c^2$$

This means that the constraint is holonomic, or *integrable*.

In general, if we can express a constraint by a relationship between any set of coordinates, then it is integrable. There are constraints for which this is not possible, such as rolling, where the constraint is expressed as a relationship between not only the coordinates but also changes in the coordinates. It is useful (for many purposes!) to be able to decide when a set of equations is integrable.

Integrable constraint always give rise to differential ones, but the converse is not true.

To see the first, consider an integrable constraint, so we have a definite relationship between the coordinates. Then we may write this relationship as the vanishing of some function,

$$f(x_i) = 0$$

In the example above, this would be

$$\begin{aligned}
f(x, y) &= y^2 - x^2 - c^2 \\
&= 0
\end{aligned}$$

Such a constraint always gives rise to a differential constraint,

$$df = \sum_{i=1}^N \frac{\partial f}{\partial x_i} dx_i = 0$$

where our system is described by N coordinates.

Conversely, however, if we have a differential constraint

$$\sum_i g_i(x_j) dx_i = 0$$

that is, one relating both the coordinates x_j and changes in the coordinates dx_i . This relation certainly remains true if we multiply by an arbitrary integrating factor, $h(x_i)$,

$$\sum_i h g_i dx_i = 0$$

Then there exists a function relating the coordinates if and only if

$$h g_i = \frac{\partial f}{\partial x_i}$$

for some function f . because then we have

$$\begin{aligned}
\sum h g_i dx_i &= \sum \frac{\partial f}{\partial x_i} dx_i \\
&= df
\end{aligned}$$

and therefore a function, $f = \int \sum h g_i dx_i$, relating the coordinates. However, such a function does not always exist, since $\int \sum h g_i dx_i$ may depend on the path of integration. The proposed function would then not be single valued, i.e., not a function. To see when the constraint is integrable (in the sense that it does give a function), we first choose a path of integration. Let the path be a curve C specified by $x_i(\lambda)$. Then we have

$$f = \sum_i \int_C h g_i \frac{dx_i}{d\lambda} d\lambda$$

Now, f is a function if and only if, its value is independent of the path of integration. This is just like the question of the existence of a Newtonian potential – equality on all curves is equivalent to the condition that the integral vanish around every closed curve,

$$\oint_{C_1-C_2} \left(\sum_i h g_i \frac{dx_i}{d\lambda} \right) d\lambda = 0$$

Thinking of the sum as a dot product in some N -dimensional space,

$$\mathbf{g} \equiv h \mathbf{g}_i$$

In three dimensions, we may immediately apply Stoke's theorem,

$$\begin{aligned} 0 &= \oint_{C_1-C_2} \left(\sum_i h g_i \frac{dx_i}{d\lambda} \right) d\lambda \\ &= \oint_{C_1-C_2} \mathbf{g} \cdot \frac{d\mathbf{x}}{d\lambda} d\lambda \\ &= \iint_S (\nabla \times \mathbf{g}) \cdot \mathbf{n} d\lambda \end{aligned}$$

and because of the freedom to choose $\mathbf{n}$ and the surface S , we must have

$$\nabla \times \mathbf{g} = 0$$

In in higher dimensions the curl is replaced by

$$\frac{\partial (h g_i)}{\partial x_j} - \frac{\partial (h g_j)}{\partial x_i} = 0$$

This result holds in any number of dimensions, and is the necessary and sufficient condition for the integrability of the differential equation

$$\sum_i g_i dx_i = 0$$

When f exists, this condition is the same as the equality of mixed partials of the function f . If we write out the components of the curl,

$$\frac{\partial (h g_i)}{\partial x_j} - \frac{\partial (h g_j)}{\partial x_i} = 0$$

we see that this is just

$$\frac{\partial f}{\partial x_j \partial x_i} - \frac{\partial f}{\partial x_i \partial x_j} = 0$$

When a constraint is integrable (holonomic), the technique of Lagrange multipliers will let us include it in a straightforward way.

5 Second law resistant problems

It is possible to ask questions which defy our usual techniques for solving the second law. The most famous example is the brachistochrone, which may be stated in the following way. Let a particle of mass m move in a given force field (for example, the gravitational field of Earth). What path between two fixed points requires the shortest travel time for the particle? To make the particle follow any given path, we allow arbitrary constraining forces as long as they do no work.

The problem is solved by finding the extremal value for the time,

$$t_{AB} = \int_A^B \frac{ds}{v(x)}$$

Before we can evaluate this integral, we must specify an entire path. Moreover, the required path is in general not a solution to the second law until we include the unknown constraint forces. The solution of this problem requires the variational techniques developed in the next Note.

6 Connection with quantum mechanics

One of the most comprehensive ways to demonstrate the classical limit of quantum mechanics is to substitute the general polar form

$$\psi(\mathbf{x}, t) = A(\mathbf{x}, t) e^{\frac{i}{\hbar} S(\mathbf{x}, t)}$$

for the wave function into the Schrödinger equation, then take the limit as Planck's constant goes to zero. The result is the Hamilton-Jacobi equation for $S(\mathbf{x}, t)$, so that in that limit, the phase of the wave function becomes the action, $S(\mathbf{x}, t) = \int L dt$