# Bertrand's Theorem

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## Circular orbits

The effective potential,

$$V_{eff} = \frac{L_\varphi^2}{2\mu r^2} + V(r)$$

has a minimum or maximum at $r_0$ if and only if

$$\begin{aligned} 0 &= \left.\frac{dV_{eff}}{dr}\right|_{r_0} \\ &= -\frac{L_\varphi^2}{\mu r_0^3} + \left.\frac{dV}{dr}\right|_{r_0} \\ &= -\frac{L_\varphi^2}{\mu r_0^3} - f(r_0) \end{aligned}$$

so we must have

$$f(r_0) = -\frac{L_\varphi^2}{\mu r_0^3}$$

At this radius, there is no net radial force, so that circular orbits are possible. Such orbits may be stable or unstable, depending on the sign of the second derivative of the effective potential. Stable orbits occur only if

$$\begin{aligned} 0 &< \left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0} \\ &= \frac{3L_\varphi^2}{\mu r_0^4} + \left.\frac{d^2V}{dr^2}\right|_{r_0} \\ &= \frac{3L_\varphi^2}{\mu r_0^4} - \frac{df}{dr}(r_0) \end{aligned}$$

and since $f(r_0) = -\frac{L_\varphi^2}{\mu r_0^3}$, we have

$$\frac{df}{dr}(r_0) < -\frac{3}{r_0} f(r_0)$$

## General case

Eliminating $\dot{\varphi} = \frac{L_\varphi}{\mu r^2}$, the remaining, radial equation of motion is

$$\mu \ddot{r} - \frac{L_\varphi^2}{\mu r^3} - f(r) = 0$$

Write this as an orbit equation, using

$$\begin{aligned}\dot{r} &= \frac{d\varphi}{dt}\frac{dr}{d\varphi} \\ &= \frac{L_\varphi}{\mu r^2}\frac{dr}{d\varphi} \\ \ddot{r} &= \frac{d\varphi}{dt}\frac{d}{d\varphi}\left(\frac{L_\varphi}{\mu r^2}\frac{dr}{d\varphi}\right) \\ &= \frac{L_\varphi}{\mu r^2}\left(-\frac{2L_\varphi}{\mu r^3}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi}{\mu r^2}\frac{d^2r}{d\varphi^2}\right) \\ &= -\frac{2L_\varphi^2}{\mu^2 r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi^2}{\mu^2 r^4}\frac{d^2r}{d\varphi^2}\end{aligned}$$

Substituting,

$$\begin{aligned}0 &= \mu\ddot{r} - \frac{L_\varphi^2}{\mu r^3} - f(r) \\ &= -\frac{2L_\varphi^2}{\mu r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi^2}{\mu r^4}\frac{d^2r}{d\varphi^2} - \frac{L_\varphi^2}{\mu r^3} - f(r)\end{aligned}$$

Now let $r = \frac{1}{u}$

$$\begin{aligned}\frac{dr}{d\varphi} &= -\frac{1}{u^2}\frac{du}{d\varphi} \\ \frac{d^2r}{d\varphi^2} &= \frac{d}{d\varphi}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right) \\ &= \frac{2}{u^3}\left(\frac{du}{d\varphi}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\varphi^2}\end{aligned}$$

The equation of motion now becomes

$$\begin{aligned}0 &= -\frac{2L_\varphi^2}{\mu r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi^2}{\mu r^4}\frac{d^2r}{d\varphi^2} - \frac{L_\varphi^2}{\mu r^3} - f(r) \\ &= -\frac{2L_\varphi^2 u^5}{\mu}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right)^2 + \frac{L_\varphi^2 u^4}{\mu}\left(\frac{2}{u^3}\left(\frac{du}{d\varphi}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\varphi^2}\right) - \frac{L_\varphi^2}{\mu r^3} - f(r) \\ &= -\frac{2L_\varphi^2 u}{\mu}\left(\frac{du}{d\varphi}\right)^2 + \frac{2L_\varphi^2 u}{\mu}\left(\frac{du}{d\varphi}\right)^2 - \frac{L_\varphi^2 u^2}{\mu}\frac{d^2u}{d\varphi^2} - \frac{L_\varphi^2 u^3}{\mu} - f(r) \\ &= -\frac{L_\varphi^2 u^2}{\mu}\frac{d^2u}{d\varphi^2} - \frac{L_\varphi^2 u^3}{\mu} - f(r)\end{aligned}$$

or

$$\frac{d^2u}{d\varphi^2} + u = -\frac{\mu}{L_\varphi^2 u^2} f\left(\frac{1}{u}\right)$$

Finally, write the force in terms of the potential,

$$\begin{aligned}f\left(\frac{1}{u}\right) &= -\frac{dV}{dr} \\ &= -\frac{du}{dr}\frac{dV}{du} \\ &= u^2\frac{dV}{du}\end{aligned}$$

Now we have simply

$$\frac{d^2u}{d\varphi^2} + u = -\frac{\mu}{L_\varphi^2}\frac{dV}{du}$$

This is the equation of a driven harmonic oscillator. This striking set of transformations is what happens when people spend 300 years working on a problem.

Now consider circular orbits. This means that $u$ does not change with $\varphi$ at all, so we have $\frac{d^2u}{d\varphi^2} = 0$ and therefore, if we set

$$h(u) = -\frac{\mu}{L_\varphi^2}\frac{dV}{du}$$

then

$$u_0 = h(u_0)$$

Next, expand $u$ and the force for small perturbations about $u_0$,

$$\begin{aligned} u &= u_0 + \eta \\ -\frac{\mu}{L_\varphi^2}\frac{dV}{du} &= h(u) \\ &= h(u_0) + h'(u_0)\eta + \frac{1}{2}h''(u_0)\eta^2 + \cdots \end{aligned}$$

Substituting these into the equation of motion,

$$\begin{aligned} \frac{d^2\eta}{d\varphi^2} + u_0 + \eta &= h(u_0) + h'(u_0)\eta + \frac{1}{2}h''(u_0)\eta^2 + \cdots \\ &= u_0 + h'(u_0)\eta + \frac{1}{2}h''(u_0)\eta^2 + \cdots \\ \frac{d^2\eta}{d\varphi^2} + [1 - h'(u_0)]\eta &= \frac{1}{2}h''(u_0)\eta^2 + \cdots \end{aligned}$$

If $1 - h'(u_0) < 0$, then the equation has exponential instead of oscillatory solutions, and the circular orbits are not stable. For stable orbits, we therefore set

$$\lambda^2 = 1 - h'(u_0) > 0$$

If we neglect the quadratic and higher terms on the right side of the equation, with initial condition $\eta = 0$ when $\varphi = 0$, we have solutions

$$\eta = A\sin\lambda\varphi$$

In order to reproduce the initial conditions after some integer number, $q$, of complete orbits, we require

$$\begin{aligned} \eta(0) &= \eta(2\pi q) \\ 0 &= A\sin 2\pi q\lambda \end{aligned}$$

so that

$$q\lambda = p$$

with $p$ another integer. We see that $\lambda$ must be rational,

$$\lambda = \frac{p}{q}$$

Assuming the force is a continuous function of position, $h'$ is continuous and $\lambda$ is also continuous. Therefore, $\lambda = \frac{p}{q}$ for *all* nearly circular orbits. Returning to the definition of $\lambda$, and using the constancy of $\lambda$,

$$\begin{aligned}
\lambda^2 &= 1 - h'(u_0) \\
&= 1 - \frac{d}{du}\left(-\frac{\mu}{L_\varphi^2 u^2} f\left(\frac{1}{u}\right)\right) \\
\lambda^2 - 1 &= -\frac{2\mu}{L_\varphi^2 u_0^3} f\left(\frac{1}{u_0}\right) + \frac{\mu}{L_\varphi^2 u^2} f'\left(\frac{1}{u_0}\right)
\end{aligned}$$

Since the circular orbit satisfies

$$\begin{aligned}
u_0 &= h(u_0) \\
&= -\frac{\mu}{L_\varphi^2 u_0^2} f\left(\frac{1}{u_0}\right) \\
f\left(\frac{1}{u_0}\right) &= -\frac{L_\varphi^2}{\mu} u_0^3
\end{aligned}$$

so that

$$\begin{aligned}
\lambda^2 - 1 &= -\frac{2\mu}{L_\varphi^2 u_0^3}\left(-\frac{L_\varphi^2}{\mu} u_0^3\right) + \frac{\mu}{L_\varphi^2 u^2} f'\left(\frac{1}{u_0}\right) \\
\lambda^2 - 1 &= 2 + \frac{\mu}{L_\varphi^2 u_0^2}\left(\frac{\left(-\frac{L_\varphi^2}{\mu} u_0^3\right)}{f\left(\frac{1}{u_0}\right)}\right) f'\left(\frac{1}{u_0}\right) \\
\lambda^2 - 3 &= -\frac{u_0}{f\left(\frac{1}{u_0}\right)} f'\left(\frac{1}{u_0}\right)
\end{aligned}$$

This must hold regardless of the value of $u_0$, so we may drop the subscript and integrate

$$\begin{aligned}
-\frac{df}{f} &= \left(\lambda^2 - 3\right)\frac{du}{u} \\
-\frac{df}{f} &= \left(\lambda^2 - 3\right)\left(-\frac{1}{r^2}\right) r dr \\
\frac{df}{f} &= \left(\lambda^2 - 3\right)\frac{dr}{r} \\
f(r) &= A r^{\lambda^2 - 3}
\end{aligned}$$

and in order to have stable, perturbatively closed orbits, the force law must be a rational power law. This give $h$ as

$$\begin{aligned}
h &= \frac{\mu}{L_\varphi^2 u^2}\frac{A}{u^{\lambda^2 - 3}} \\
&= \frac{\mu A}{L_\varphi^2} u^{1-\lambda^2}
\end{aligned}$$

Now return to the full equation of motion

$$\frac{d^2\eta}{d\varphi^2} + u_0 + \eta = u_0 + h'(u_0)\,\eta + \frac{1}{2}h''(u_0)\,\eta^2 + \frac{1}{6}h'''(u_0)\,\eta^3 + \cdots$$

and expand in a Fourier series,

$$\eta(\varphi) = \eta_0 + \eta_1 \cos \lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots$$

We also need powers of $\eta$. Keeping up to third order,

$$\begin{aligned}
(\eta_0 + \eta_1 \cos \lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots)^2 &= \eta_0^2 + \eta_0\eta_1 \cos \lambda\varphi + \eta_0\eta_2 \cos 2\lambda\varphi + \eta_0\eta_3 \cos 3\lambda\varphi \\
&\quad + \eta_0\eta_1 \cos \lambda\varphi + \eta_1^2 \cos^2 \lambda\varphi + \eta_2\eta_1 \cos \lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0\eta_2 \cos 2\lambda\varphi + \eta_1\eta_2 \cos 2\lambda\varphi \cos \lambda\varphi + \eta_0\eta_3 \cos 3\lambda\varphi \\
(\eta_0 + \eta_1 \cos \lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots)^3 &= \eta_0^3\eta_0 + \eta_0^2\eta_1 \cos \lambda\varphi + \eta_0^2\eta_2 \cos 2\lambda\varphi + \eta_0^2\eta_3 \cos 3\lambda\varphi + \cdots \\
&\quad + \eta_0^2\eta_1 \cos \lambda\varphi + \eta_0\eta_1^2 \cos^2 \lambda\varphi + \eta_2\eta_0\eta_1 \cos \lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0\eta_0^2\eta_2 \cos 2\lambda\varphi + \eta_1\eta_0\eta_2 \cos 2\lambda\varphi \cos \lambda\varphi \\
&\quad + \eta_0^2\eta_3 \cos 3\lambda\varphi \\
&\quad + \eta_0^2\eta_1 \cos \lambda\varphi + \eta_0\eta_1^2 \cos^2 \lambda\varphi + \eta_0\eta_1\eta_2 \cos \lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0\eta_1^2 \cos^2 \lambda\varphi + \eta_1^3 \cos^3 \lambda\varphi \\
&\quad + \eta_0\eta_2\eta_1 \cos \lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0^2\eta_2 \cos 2\lambda\varphi + \eta_0\eta_1\eta_2 \cos 2\lambda\varphi \cos \lambda\varphi \\
&\quad + \eta_0\eta_1\eta_2 \cos 2\lambda\varphi \cos \lambda\varphi \\
&\quad + \eta_0^2\eta_3 \cos 3\lambda\varphi
\end{aligned}$$

and using addition formulas

$$\begin{aligned}
\cos^2 \lambda\varphi &= \frac{1}{2}(1 + \cos 2\lambda\varphi) \\
\cos \lambda\varphi \cos 2\lambda\varphi &= \frac{1}{2}[\cos(\lambda\varphi + 2\lambda\varphi) + \cos(\lambda\varphi - 2\lambda\varphi)] \\
&= \frac{1}{2}[\cos(3\lambda\varphi) + \cos(\lambda\varphi)] \\
\cos^3 \lambda\varphi &= \cos \lambda\varphi \frac{1}{2}(1 + \cos 2\lambda\varphi) \\
&= \frac{1}{2}\cos \lambda\varphi + \frac{1}{4}\cos(3\lambda\varphi) + \frac{1}{4}\cos(\lambda\varphi) \\
&= \frac{3}{4}\cos \lambda\varphi + \frac{1}{4}\cos(3\lambda\varphi)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\eta_0 + \eta_1 \cos \lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots)^2 &= \eta_0^2 + \frac{1}{2}\eta_1^2 + (2\eta_0\eta_1 + \eta_1\eta_2)\cos(\lambda\varphi) \\
&\quad + \left(2\eta_0\eta_2 + \frac{1}{2}\eta_1^2\right)\cos 2\lambda\varphi + (2\eta_0\eta_3 + \eta_1\eta_2)\cos 3\lambda\varphi \\
(\eta_0 + \eta_1 \cos \lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots)^3 &= \eta_0^3 + \left(3\eta_0^2\eta_1\right)\cos \lambda\varphi + \left(3\eta_0^2\eta_2\right)\cos 2\lambda\varphi + \left(3\eta_0^2\eta_3\right)\cos 3\lambda\varphi \\
&\quad + 3\eta_0\eta_1^2 \cos^2 \lambda\varphi + 6\eta_0\eta_1\eta_2 \cos 2\lambda\varphi \cos \lambda\varphi + \eta_1^3 \cos^3 \lambda\varphi \\
&= \eta_0^3 + \left(3\eta_0^2\eta_1\right)\cos \lambda\varphi + \left(3\eta_0^2\eta_2\right)\cos 2\lambda\varphi + \left(3\eta_0^2\eta_3\right)\cos 3\lambda\varphi \\
&\quad + 3\eta_0\eta_1^2\frac{1}{2}(1 + \cos 2\lambda\varphi) + 6\eta_0\eta_1\eta_2\frac{1}{2}[\cos(3\lambda\varphi) + \cos(\lambda\varphi)] \\
&\quad + \eta_1^3\frac{3}{4}\cos \lambda\varphi + \frac{1}{4}\eta_1^3\cos(3\lambda\varphi) \\
&= \eta_0^3 + \frac{3}{2}\eta_0\eta_1^2 + \left(3\eta_0^2\eta_1 + 3\eta_0\eta_1\eta_2 + \frac{3}{4}\eta_1^3\right)\cos \lambda\varphi
\end{aligned}$$

$$+\left(3\eta_0^2\eta_2+\frac{3}{2}\eta_0\eta_1^2\right)\cos 2\lambda\varphi+\left(3\eta_0^2\eta_3+\frac{1}{4}\eta_1^3+3\eta_0\eta_1\eta_2\right)\cos 3\lambda\varphi$$

Substituting and expanding,

$$\begin{aligned}
\frac{d^2\eta}{d\varphi^2}+u_0+\eta &= u_0+h'(u_0)\,\eta+\frac{1}{2}h''(u_0)\,\eta^2+\frac{1}{6}h'''(u_0)\,\eta^3+\cdots \\
\left(-\eta_1\lambda^2\cos\lambda\varphi-4\lambda^2\eta_2\cos 2\lambda\varphi-9\lambda^2\eta_3\cos 3\lambda\varphi+\cdots\right)+\eta &= h'(u_0)\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right) \\
&\quad+\frac{1}{2}h''(u_0)\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right)^2 \\
&\quad+\frac{1}{6}h'''(u_0)\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right)^3
\end{aligned}$$

Equating like terms,

$$\begin{aligned}
\eta_0 &= h'(u_0)\,\eta_0+\frac{1}{2}h''(u_0)\,\eta_0^2+\frac{1}{2}h''(u_0)\,\frac{1}{2}\eta_1^2+\frac{1}{6}h'''(u_0)\,\eta_0^3+\frac{1}{6}h'''(u_0)\,\frac{3}{2}\eta_0\eta_1^2 \\
\left(-\eta_1\lambda^2+\eta_1\right)\cos\lambda\varphi &= \left[h'(u_0)\,\eta_1+\frac{1}{2}h''(u_0)\left(2\eta_0\eta_1+\eta_1\eta_2\right)+\frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_1+3\eta_0\eta_1\eta_2+\frac{3}{4}\eta_1^3\right)\right]\cos\lambda\varphi \\
-4\lambda^2\eta_2\cos 2\lambda\varphi+\eta_2\cos 2\lambda\varphi &= \left[h'(u_0)\,\eta_2+\frac{1}{2}h''(u_0)\left(2\eta_0\eta_2+\frac{1}{2}\eta_1^2\right)+\frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_2+\frac{3}{2}\eta_0\eta_1^2\right)\right]\cos 2\lambda\varphi \\
-9\lambda^2\eta_3\cos 3\lambda\varphi+\eta_3\cos 3\lambda\varphi &= \left[h'(u_0)\,\eta_3+\frac{1}{2}h''(u_0)\left(2\eta_0\eta_3+\eta_1\eta_2\right)+\frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_3+\frac{1}{4}\eta_1^3+3\eta_0\eta_1\eta_2\right)\right]\cos 3\lambda\varphi
\end{aligned}$$

Now we keep only the lowest order terms from each equation, i.e., we have $\eta_1 \gg \eta_1\eta_0$ and so on. We set $h'(u_0)=1-\lambda^2$ and solve

$$\begin{aligned}
0 &= -\lambda^2\eta_0+\frac{1}{2}h''(u_0)\,\eta_0^2+\frac{1}{6}h'''(u_0)\,\eta_0^3+\left(\frac{1}{4}h'''(u_0)\,\eta_0+\frac{1}{4}h''(u_0)\right)\eta_1^2 \\
0 &= -\lambda^2\eta_0+\frac{1}{4}h''(u_0)\,\eta_1^2 \\
\eta_0 &= \frac{1}{4\lambda^2}h''(u_0)\,\eta_1^2 \\
0 &= \frac{1}{2}h''(u_0)\left(2\eta_0\eta_1+\eta_1\eta_2\right)+\frac{1}{8}\eta_1^3h'''(u_0) \\
&= \frac{1}{2}h''(u_0)\left(\frac{1}{2\lambda^2}h''(u_0)\,\eta_1^3-\frac{1}{12\lambda^2}h''(u_0)\,\eta_1^3\right)+\frac{1}{8}\eta_1^3h'''(u_0) \\
&= \eta_1^3\frac{1}{24\lambda^2}\left(5\left(h''(u_0)\right)^2+3\lambda^2h'''(u_0)\right) \\
\eta_2 &= -\frac{1}{12\lambda^2}h''(u_0)\,\eta_1^2 \\
\eta_3 &= -\frac{1}{8\lambda^2}\left[\frac{1}{2}h''(u_0)\,\eta_1\eta_2+\frac{1}{24}\eta_1^3h'''(u_0)\right]
\end{aligned}$$

or, summarizing,

$$\begin{aligned}
\eta_0 &= \frac{1}{4\lambda^2}h''(u_0)\,\eta_1^2 \\
0 &= \frac{1}{24\lambda^2}\eta_1^3\left(5\left(h''(u_0)\right)^2+3\lambda^2h'''(u_0)\right) \\
\eta_2 &= -\frac{1}{12\lambda^2}h''(u_0)\,\eta_1^2 \\
\eta_3 &= -\frac{1}{8\lambda^2}\left[\frac{1}{2}h''(u_0)\,\eta_1\eta_2+\frac{1}{24}\eta_1^3h'''(u_0)\right]
\end{aligned}$$

where $\eta_1$ is constant. Now, using

$$h = \frac{\mu A}{L_\varphi^2} u^{1-\lambda^2}$$

we may find the derivatives,

$$\begin{aligned}
1-\lambda^2 = h'(u_0) &= \left(1-\lambda^2\right) \frac{\mu A}{L_\varphi^2} u_0^{-\lambda^2} \\
\frac{\mu A}{L_\varphi^2} u_0^{-\lambda^2} &= 1 \\
h''(u_0) &= -\lambda^2 \left(1-\lambda^2\right) \frac{\mu A}{L_\varphi^2} u_0^{-\lambda^2-1} \\
&= -\frac{\lambda^2 \left(1-\lambda^2\right)}{u_0} \\
h'''(u_0) &= \frac{\left(1+\lambda^2\right) \lambda^2 \left(1-\lambda^2\right)}{u_0^2}
\end{aligned}$$

so we may solve the equality

$$\begin{aligned}
0 &= \eta_1^3 \frac{1}{24\lambda^2} \left(5 \left(h''(u_0)\right)^2 + 3\lambda^2 h'''(u_0)\right) \\
0 &= \frac{1}{\lambda^2} \left(\frac{5\lambda^4 \left(1-\lambda^2\right)^2}{u_0^2} + 3\lambda^2 \frac{\left(1+\lambda^2\right) \lambda^2 \left(1-\lambda^2\right)}{u_0^2}\right) \\
0 &= 5\lambda^2 \left(1-\lambda^2\right)^2 + 3\lambda^2 \left(1+\lambda^2\right) \left(1-\lambda^2\right) \\
&= 8\lambda^2 - 10\lambda^4 + 2\lambda^6 \\
&= 2\lambda^2 \left(1-\lambda^2\right) \left(4-\lambda^2\right)
\end{aligned}$$

and we see that the only candidate power laws, $f(r) = Ar^{\lambda^2-3}$, for stable, closed orbits are:

$$\begin{aligned}
f(r) &= Ar^{-3} \\
f(r) &= Ar^{-2} \\
f(r) &= Ar
\end{aligned}$$

The first of these yields only perfectly circular orbits, so the only nontrivial cases are the inverse square law and Hookes' law.

This condition only shows that these power laws are necessary. That they are sufficient to produce closed orbits requires solving for their orbits exactly.