Two dimensional oscillator and central forces

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1 Hooke's law in two dimensions

Consider a radial Hooke's law force in 2-dimensions,

$$\mathbf{F} = -kr\hat{\mathbf{r}}$$

where the force is along the radial unit vector $\hat{\mathbf{r}}$ and depends on the distance from the origin, r, where

$$\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{i}}\cos\varphi + \hat{\mathbf{j}}\sin\varphi \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

and therefore

$$\begin{aligned} \mathbf{r} &= r\hat{\mathbf{r}} \\ &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \end{aligned}$$

Let the initial position and velocity (at $t_0 = 0$) be

$$\begin{aligned} \mathbf{x}(0) &= x_0\hat{\mathbf{i}} \\ \mathbf{v}(0) &= v_0\hat{\mathbf{j}} \end{aligned}$$

Find the motion, $\mathbf{x}(t)$, and determine whether this initial condition is sufficiently general.

2 Solution in Cartesian coordinates

In Cartesian coordinates, the solution is immediate. Writing the force and acceleration as

$$\begin{aligned} \mathbf{F} &= -k\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right) \\ \mathbf{a} &= \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} \end{aligned}$$

where the overdots denote time derivatives, e.g., $\ddot{x} \equiv \frac{d^2x}{dt^2}$, we have

$$-k\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right) = m\left(\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}}\right)$$

Defining the frequency of oscillation by $\omega^2 = \frac{k}{m}$, the equation decouples into two simple harmonic oscillators,

$$\begin{aligned} m\ddot{x} + \omega^2 x &= 0 \\ m\ddot{y} + \omega^2 y &= 0 \end{aligned}$$

with the immediate solution

$$
\begin{aligned}
x &= a\cos\omega t + b\sin\omega t \\
&= A\cos\omega\left(t-t_0\right) \\
y &= C\cos\omega t + D\sin\omega t
\end{aligned}
$$

Since the force diverges as the motion moves off to infinity, we expect the motion to be bounded. We can check this with the energy theorem,

$$
\begin{aligned}
\int_{x_0}^{x} \mathbf{F}\cdot d\mathbf{x} &= \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{v}_0^2 \\
-k\int_{\mathbf{x}_0}^{\mathbf{x}} \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right)\cdot\left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}\right) &= \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{v}_0^2 \\
-k\int_{\mathbf{x}_0}^{\mathbf{x}} \left(xdx + ydy\right) &= \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{v}_0^2 \\
-k\left(\int_{x_0}^{x} xdx + \int_{y_0}^{y} ydy\right) &= \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{v}_0^2 \\
-k\left(x^2+y^2\right) + k\left(x_0^2+y_0^2\right) &= \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m\mathbf{v}_0^2
\end{aligned}
$$

and therefore

$$
\begin{aligned}
E &= \frac{1}{2}m\mathbf{v}_0^2 + \frac{1}{2}k\mathbf{x}_0^2 \\
&= \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}k\mathbf{x}^2
\end{aligned}
$$

remains constant. For finite initial conditions, E is finite, so both the velocity and position remain bounded at all times,

$$
\begin{aligned}
|\mathbf{x}| &\leq \frac{2E}{k} \\
|\mathbf{v}| &\leq \frac{2E}{m}
\end{aligned}
$$

This means there must be turning points for $\mathbf{x}$, i.e., points where a positive velocity $\dot{\mathbf{x}}$ decreases to zero as $|\mathbf{x}|$ reaches a maximum $|\mathbf{x}_{max}|$ begins to decrease. Notice also that the velocity cannot be zero for any position less than the maximum if E is nonzero.

These observations allow us to choose the initial conditions so that $\dot{\mathbf{x}} = 0$, and we may rotate the xy-axes until x-axis lies in the direction of the maximum vector $\mathbf{x}_{max}$. This means the initial $x_0 = x_{max}$ and $\dot{x}_0 = 0$. Since $\mathbf{x}$ is in the x-direction, $y_0 = 0$, and from the conservation of energy $\dot{y}_0 = \sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx_{max}^2\right)} \equiv v_{min}$. Choosing the initial time to be $t_0 = 0$ so that $x(0) = x_{max}, \dot{x}(0) = 0$, we have

$$
\begin{aligned}
x &= x_{max}\cos\omega t \\
y &= \frac{v_{min}}{\omega}\sin\omega t
\end{aligned}
$$

The shape of the orbit is an ellipse, since

$$
\frac{x^2}{x_{max}^2} + \frac{y^2}{\left(v_{min}/\omega\right)^2} = 1
$$

3 Solution as a central force problem

We may treat the problem in polar coordinates instead. This introduces complications which are unnecessary here, but which illustrate several of the general techniques used for the Kepler problem, or other central forces, $\mathbf{F} = -f(r)\,\hat{\mathbf{r}}$.

We begin with the energy theorem. Taking the dot product with $\hat{\mathbf{r}}$, we have

$$\begin{aligned}
\mathbf{F} &= -kr\hat{\mathbf{r}} \\
\mathbf{F}\cdot\hat{\mathbf{r}} &= -kr
\end{aligned}$$

To substitute into the second law, we need the acceleration in polar coordinates:

$$\begin{aligned}
\mathbf{r} &= r\hat{\mathbf{r}} \\
\mathbf{v} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}
\end{aligned}$$

Now, with

$$\hat{\mathbf{r}} = \hat{\mathbf{i}}\cos\varphi + \hat{\mathbf{j}}\sin\varphi$$

we have

$$\begin{aligned}
\frac{d\hat{\mathbf{r}}}{dt} &= \left(-\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi\right)\frac{d\varphi}{dt} \\
&= \hat{\boldsymbol{\varphi}}\dot{\varphi}
\end{aligned}$$

so that

$$\begin{aligned}
\mathbf{v} &= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \\
&= \dot{r}\hat{\mathbf{r}} + r\dot{\varphi}\hat{\boldsymbol{\varphi}}
\end{aligned}$$

The acceleration is then,

$$\begin{aligned}
\mathbf{a} &= \frac{d}{dt}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\varphi}\hat{\boldsymbol{\varphi}}\right) \\
&= \left(\ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\varphi}\hat{\boldsymbol{\varphi}} + \dot{r}\dot{\varphi}\hat{\boldsymbol{\varphi}} + r\ddot{\varphi}\hat{\boldsymbol{\varphi}} + r\dot{\varphi}\frac{d}{dt}\hat{\boldsymbol{\varphi}}\right)
\end{aligned}$$

Then, using

$$\begin{aligned}
\frac{d}{dt}\hat{\boldsymbol{\varphi}} &= \frac{d}{dt}\left(-\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi\right) \\
&= \dot{\varphi}\left(-\hat{\mathbf{i}}\cos\varphi - \hat{\mathbf{j}}\sin\varphi\right) \\
&= -\dot{\varphi}\hat{\mathbf{r}}
\end{aligned}$$

the acceleration becomes

$$\begin{aligned}
\mathbf{a} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\varphi}\hat{\boldsymbol{\varphi}} + \dot{r}\dot{\varphi}\hat{\boldsymbol{\varphi}} + r\ddot{\varphi}\hat{\boldsymbol{\varphi}} - r\dot{\varphi}^2\hat{\mathbf{r}} \\
&= \left(\ddot{r} - r\dot{\varphi}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\hat{\boldsymbol{\varphi}}
\end{aligned}$$

Now Newton's second law gives

$$-kr\hat{\mathbf{r}} = m\left(\ddot{r} - r\dot{\varphi}^2\right)\hat{\mathbf{r}} + m\left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\hat{\boldsymbol{\varphi}}$$

so we have two independent equations,

$$\begin{aligned}
-kr &= m\left(\ddot{r} - r\dot{\varphi}^2\right) \\
0 &= m\left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)
\end{aligned}$$

Define $\omega^2 \equiv \frac{k}{m}$. We can integrate the second immediately:

$$\begin{aligned}
2\dot{r}\dot{\varphi} &= -r\ddot{\varphi} \\
\frac{2}{r}\frac{dr}{dt} &= -\frac{1}{\dot{\varphi}}\frac{d\dot{\varphi}}{dt} \\
\int_{r_0}^{r} \frac{2}{r}dr &= -\int_{\dot{\varphi}_0}^{\dot{\varphi}} \frac{1}{\dot{\varphi}}d\dot{\varphi} \\
\ln\frac{r^2}{r_0^2} &= -\ln\frac{\dot{\varphi}}{\dot{\varphi}_0} \\
\ln\frac{r^2}{r_0^2} &= \ln\frac{\dot{\varphi}_0}{\dot{\varphi}} \\
r^2\dot{\varphi} &= r_0^2\dot{\varphi}_0
\end{aligned}$$

Recognizing $L = mr^2\dot{\varphi}$, we see that this is conservation of angular momentum. We might have seen the constant more readily by multiplying the original equation by r,

$$\begin{aligned}
0 &= m\left(2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi}\right) \\
&= \frac{d}{dt}\left(mr^2\dot{\varphi}\right)
\end{aligned}$$

Lagrangian techniques will let us find results like this quickly.

Setting $\dot{\varphi} = \frac{L}{mr^2}$ with L constant, the first equation now becomes

$$\begin{aligned}
-\omega^2 r &= \left(\ddot{r} - r\left(\frac{L}{mr^2}\right)^2\right) \\
-\omega^2 r &= \ddot{r} - \frac{L^2}{m^2r^3} \\
0 &= \ddot{r} + \omega^2 r - \frac{L^2}{m^2r^3}
\end{aligned}$$

The energy theorem gives us the first integral. Multiplying by $\dot{r}$

$$\begin{aligned}
0 &= \dot{r}\ddot{r} + \omega^2 r\dot{r} - \frac{L^2}{m^2r^3}\dot{r} \\
&= \dot{r}\frac{d\dot{r}}{dt} + \omega^2 r\frac{dr}{dt} - \frac{L^2}{m^2r^3}\frac{dr}{dt}
\end{aligned}$$

then by dt,

$$\begin{aligned}
\dot{r}d\dot{r} &= -\left(\omega^2 r - \frac{L^2}{m^2r^3}\right)dr \\
\int_{\dot{r}_0}^{\dot{r}} \dot{r}d\dot{r} &= -\int_{r_0}^{r}\left(\omega^2 r - \frac{L^2}{m^2r^3}\right)dr \\
\frac{1}{2}\left(\dot{r}^2 - \dot{r}_0^2\right) &= -\frac{\omega^2}{2}\left(r^2 - r_0^2\right) - \left(\frac{L^2}{2m^2r^2} - \frac{L^2}{2m^2r_0^2}\right) \\
\frac{1}{2}\left(\dot{r}^2 + \frac{L^2}{m^2r^2} + \omega^2 r^2\right) &= \frac{1}{2}\left(\dot{r}_0^2 + \omega^2 r_0^2 + \frac{L^2}{m^2r_0^2}\right) \equiv \frac{E}{m}
\end{aligned}$$

Now solve for the radial velocity, $\dot{r}$,

$$\begin{aligned} \dot{r}^2 + \frac{L^2}{m^2r^2} + \omega^2 r^2 &= \frac{2E}{m} \\ \frac{dr}{dt} &= \sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} - \omega^2 r^2} \end{aligned}$$

We can integrate this to find $r(t)$, but it is more instructive to find the shape of the spatial orbit, $r(\varphi)$. To do this, divide the whole equation by $\dot{\varphi}$. Since $\frac{dr/dt}{d\varphi/dt} = \frac{dr}{d\varphi}$, we get

$$\begin{aligned} \frac{dr}{d\varphi} &= \frac{1}{\dot{\varphi}}\sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} - \frac{k}{m}r^2} \\ &= \frac{mr^2}{L}\sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} - \frac{k}{m}r^2} \\ &= r\sqrt{-1 + \frac{2mE}{L^2}r^2 - \frac{m^2\omega^2}{L^2}r^4} \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\varphi} d\varphi &= \int_{r_0}^{r} \frac{dr}{r\sqrt{-1 + \frac{2mE}{L^2}r^2 - \frac{m^2\omega^2}{L^2}r^4}} \\ \varphi &= \int_{r_0}^{r} \frac{dr}{r\sqrt{-1 + \frac{2mE}{L^2}r^2 - \frac{m^2\omega^2}{L^2}r^4}} \end{aligned}$$

Let $r = \frac{1}{u}$, so $dr = -\frac{du}{u^2}$ and

$$\begin{aligned} \varphi &= -\int_{r_0}^{r} \frac{du}{u\sqrt{-1 + \frac{2mE}{L^2u^2} - \frac{m^2\omega^2}{L^2u^4}}} \\ &= -\int_{r_0}^{r} \frac{udu}{\sqrt{-u^4 + \frac{2mE}{L^2}u^2 - \frac{m^2\omega^2}{L^2}}} \end{aligned}$$

We leave the limits in terms of r and substitute back later. Now with $y = u^2$,

$$\varphi = -\frac{1}{2}\int_{r_0}^{r} \frac{dy}{\sqrt{-y^2 + \frac{2mE}{L^2}y - \frac{m^2\omega^2}{L^2}}}$$

where we leave the limits in terms of u to substitute back later, and integrate by completing the square in the denominator.

$$\begin{aligned} -y^2 + \frac{2mE}{L^2}y - \frac{m^2\omega^2}{L^2} &= -\left(y - \frac{mE}{L^2}\right)^2 + \frac{m^2E^2}{L^4} - \frac{m^2\omega^2}{L^2} \\ &= \frac{m^2E^2 - m^2\omega^2L^2}{L^4} - \left(y - \frac{mE}{L^2}\right)^2 \end{aligned}$$

Then setting $z = y - \frac{mE}{L^2}$,

$$\varphi = -\frac{1}{2}\int_{r_0}^{r} \frac{dz}{\sqrt{\frac{m^2E^2 - m^2\omega^2L^2}{L^4} - z^2}}$$

$$= \quad -\frac{1}{2}\frac{L^2}{\sqrt{m^2E^2 - m^2\omega^2L^2}}\int\limits_{r_0}^{r}\frac{dz}{\sqrt{1-\frac{L^4z^2}{m^2E^2-m^2\omega^2L^2}}}$$

Now substitute

$$\begin{aligned}
\sin\theta &= \frac{L^2 z}{\sqrt{m^2E^2 - m^2\omega^2L^2}} \\
\cos\theta d\theta &= \frac{L^2 dz}{\sqrt{m^2E^2 - m^2\omega^2L^2}}
\end{aligned}$$

so that

$$\begin{aligned}
\varphi &= -\frac{1}{2}\frac{L^2}{\sqrt{m^2E^2 - m^2\omega^2L^2}}\int\limits_{r_0}^{r}\frac{dz}{\sqrt{1-\frac{L^4z^2}{m^2E^2-m^2\omega^2L^2}}} \\
&= -\frac{1}{2}\int\limits_{r_0}^{r} d\theta \\
&= -\left.\frac{\theta}{2}\right|_{r_0}^{r} \\
&= -\left.\frac{1}{2}\sin^{-1}\frac{L^2 z}{\sqrt{m^2E^2 - m^2\omega^2L^2}}\right|_{r_0}^{r} \\
&= -\left.\frac{1}{2}\sin^{-1}\frac{L^2\left(y-\frac{mE}{L^2}\right)}{\sqrt{m^2E^2 - m^2\omega^2L^2}}\right|_{r_0}^{r} \\
&= -\left.\frac{1}{2}\sin^{-1}\frac{L^2\left(u^2-\frac{mE}{L^2}\right)}{\sqrt{m^2E^2 - m^2\omega^2L^2}}\right|_{r_0}^{r} \\
&= -\frac{1}{2}\sin^{-1}\frac{L^2\left(\frac{1}{r^2}-\frac{mE}{L^2}\right)}{\sqrt{m^2E^2 - m^2\omega^2L^2}} + \frac{1}{2}\sin^{-1}\frac{L^2\left(\frac{1}{r_0^2}-\frac{mE}{L^2}\right)}{\sqrt{m^2E^2 - m^2\omega^2L^2}}
\end{aligned}$$

Define

$$\varphi_0 \equiv \frac{1}{2}\sin^{-1}\frac{L^2\left(\frac{1}{r_0^2}-\frac{mE}{L^2}\right)}{\sqrt{m^2E^2 - m^2\omega^2L^2}}$$

so that

$$\begin{aligned}
\varphi &= -\frac{1}{2}\sin^{-1}\frac{L^2\left(\frac{1}{r^2}-\frac{mE}{L^2}\right)}{\sqrt{m^2E^2 - m^2\omega^2L^2}} + \varphi_0 \\
2\left(\varphi-\varphi_0\right) &= \sin^{-1}\frac{L^2\left(\frac{mE}{L^2}-\frac{1}{r^2}\right)}{\sqrt{m^2E^2 - m^2\omega^2L^2}} \\
\sin 2\left(\varphi-\varphi_0\right) &= \frac{L^2\left(\frac{mE}{L^2}-\frac{1}{r^2}\right)}{m\sqrt{E^2-\omega^2L^2}} \\
&= \frac{L^2\left(\frac{mE}{L^2}-\frac{1}{r^2}\right)}{m\sqrt{E^2-\omega^2L^2}}
\end{aligned}$$

This gives the equation of the orbit,

$$\sqrt{E^2-\omega^2L^2}\sin 2\left(\varphi-\varphi_0\right) \quad = \quad E - \frac{L^2}{mr^2}$$

$$
\begin{aligned}
\frac{L^2}{mr^2} &= E - \sqrt{E^2 - \omega^2 L^2} \sin 2\left(\varphi - \varphi_0\right) \\
r^2 &= \frac{L^2}{mE} \frac{1}{1 - \sqrt{1 - \frac{\omega^2 L^2}{E^2}} \sin 2\left(\varphi - \varphi_0\right)}
\end{aligned}
$$

Compare familiar equations for an ellipse:

$$
\begin{aligned}
r &= \frac{A}{1 + \varepsilon \sin\theta} \\
r^2 &= \frac{B}{1 + 2\varepsilon \sin\theta + \varepsilon^2 \sin^2\theta}
\end{aligned}
$$

We have

$$
\begin{aligned}
r^2 &= \frac{R^2}{1 - k \sin 2\left(\varphi - \varphi_0\right)} \\
r^2 - 2kr^2 \sin\varphi \cos\varphi &= R^2 \\
x^2 + y^2 - 2kxy &= R^2 \\
x'^2 + y'^2 - 2k\left(x' \cos\varphi + y' \sin\varphi\right)\left(y' \cos\varphi - x' \sin\varphi\right) &= R^2 \\
x'^2 + y'^2 - 2k\left(y'x' \cos^2\varphi - x'^2 \cos\varphi \sin\varphi + y'^2 \sin\varphi \cos\varphi - x'y' \sin^2\varphi\right) &= R^2
\end{aligned}
$$

Let $\varphi = \frac{\pi}{4}$,

$$
\begin{aligned}
x'^2 + y'^2 - 2k\left(-\frac{1}{2}x'^2 + \frac{1}{2}y'^2\right) &= R^2 \\
\left(1 + k\right) x'^2 + \left(1 - k\right) y'^2 &= R^2
\end{aligned}
$$

and this is of the same form as a general ellipse,

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
$$