Brachistochrone

October 2, 2012

1 Statement of the problem

We consider particle of mass m a path through Earth (mass, M, radius R, nonrotating, uniform density), starting at the surface at rest. The path ends at another fixed point on the surface at time $t\,[r\,(s)\,,\theta\,(s)\,,\varphi\,(s)]$, which is a functional of the path. We wish to find the path which minimizes t. In spherical coordinates, we have the time of flight

$$t = \int \frac{ds}{v}$$

Notice that the solution path to this problem is *not* a solution to Newton's second law. We constrain the particle to follow a path $(r,\theta,\varphi) = (r\,(\lambda)\,,\theta\,(\lambda)\,,\varphi\,(\lambda))$, where λ is an arbitrary parameter. We ultimately take $\lambda = s$ to be arc-length, and assume that the path is frictionless. This means that any force exerted by the path is orthogonal to the line of motion, and therefore does not do any work. The constraint forces do not change the energy of the particle. It is this assumption that lets us use Newtonian mechanics to find the velocity of the particle.

Since the only force is gravity, and gravity acts only in the radial direction, we have rotational symmetry about the center of Earth. This means we can place the initial position at any value we choose for the angular coordinates. Therefore, for the initial position at $\lambda = 0$, we may place the particle on the equator at $\varphi = 0$:

$$(r\,(0)\,,\theta\,(0)\,,\varphi\,(0)) = \left(R, \frac{\pi}{2}, 0\right)$$

The particle starts from rest, so we have the initial rates of change,

$$
\begin{aligned}
\left(\frac{dr}{d\lambda}\,(0)\,, \frac{d\theta}{d\lambda}\,(0)\,, \frac{d\varphi}{d\lambda}\,(0)\right) &= \left(\dot{r}\,(0)\,, \dot{\theta}\,(0)\,, \dot{\varphi}\,(0)\right) \\
&= (0,0,0)
\end{aligned}
$$

This makes it clear that the solution is not a solution to Newton's second law without including the constraint forces, because a particle with these initial conditions would otherwise fall straight to the center of Earth. This means that the ordinary angular momentum of the particle ($l_\varphi = mr^2\dot{\varphi}$) must change.

1.1 The potential energy

For a particle in Earth's gravitational field, the force is

$$
\begin{aligned}
\mathbf{F} &= -\frac{GM_E\,(r)\,m}{r^2}\hat{\mathbf{r}} \\
&= \begin{cases} -\frac{4\pi G\rho m}{3}r\hat{\mathbf{r}} & r < R \\ -\frac{GMm}{r^2}\hat{\mathbf{r}} & r > R \end{cases}
\end{aligned}
$$

1

where, since $M = \frac{4\pi}{3}R^3\rho$, we may write $\mathbf{F} = -\frac{GMm}{R^3}r\hat{\mathbf{r}}$ for $r < R$. Taking the zero of the potential at infinity (this is arbitrary; any point will do),

$$V = -\int_{\infty}^{r} F\,dr$$

Outside Earth, $V = -\frac{GMm}{r}$ as usual. Inside Earth, we may divide the integral to get

$$
\begin{aligned}
V &= -\int_{\infty}^{r} F\,dr \\
&= -\int_{\infty}^{R} F\,dr - \int_{R}^{r} F\,dr \\
&= -\frac{GMm}{R} + \frac{GMm}{R^3}\int_{R}^{r} r\,dr \\
&= -\frac{GMm}{R} - \frac{GMm}{2R^3}\left(R^2 - r^2\right)
\end{aligned}
$$

The potential is continuous, vanishes at infinity, and reaches its most negative value at the origin.

1.2 The Lagrangian and conserved quantities

In spherical coordinates the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$

where the overdot here means $\frac{d}{dt}$. Therefore, inside,

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + \frac{GMm}{R} + \frac{GMm}{2R^3}\left(R^2 - r^2\right)$$

There is no explicit time dependence of the Lagrangian, so we have a conserved energy,

$$
\begin{aligned}
E &= \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \\
&= \frac{\partial L}{\partial \dot{r}}\dot{r} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi} - L \\
&= m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) - L \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) - \frac{GMm}{R} - \frac{GMm}{2R^3}\left(R^2 - r^2\right)
\end{aligned}
$$

The speed along any curve is therefore

$$
\begin{aligned}
v &= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2} \\
&= \sqrt{\frac{2E}{m} + \frac{2GM}{R} + \frac{GM}{R^3}\left(R^2 - r^2\right)}
\end{aligned}
$$

Since the particle starts from rest at $r = R$, we can evaluate the energy there by setting $r = R$,

$$E = -\frac{GMm}{R}$$

2

so we have

$$
\begin{aligned}
v &= \sqrt{-\frac{2GM}{R} + \frac{2GM}{R} + \frac{GM}{R^3}\left(R^2 - r^2\right)} \\
&= \sqrt{\frac{GM}{R^3}\left(R^2 - r^2\right)} \\
&= \sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}
\end{aligned}
$$

where $g = \frac{GM}{R^2}$ is the usual surface acceleration. The time is

$$
\begin{aligned}
t &= \int \frac{ds}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} \\
&= \int \frac{\sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} \\
&= \int \frac{\sqrt{\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 + r^2 \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} d\lambda
\end{aligned}
$$

After variation, we will set $\lambda = s$. It will be convenient from here on to change our notation and let the overdot denote $\frac{d}{d\lambda} \longrightarrow \frac{d}{ds}$ instead of $\frac{d}{dt}$, and from here on always write $\frac{d}{dt}$ explicitly for time derivatives. These are related by,

$$
\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v\frac{d}{ds}
$$

though we will not need this. Employing this change, we may write the time functional as

$$
t\left[r\left(\lambda\right), \theta\left(\lambda\right), \varphi\left(\lambda\right)\right] = \int \frac{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} d\lambda
$$

We get a second constant of motion by noting that φ is cyclic, so that

$$
\begin{aligned}
L_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} \\
&= \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} \frac{\partial}{\partial \dot{\varphi}}\left(\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2}\right) \\
&= \frac{r^2 \sin^2\theta\dot{\varphi}}{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2}\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}
\end{aligned}
$$

is constant, and when we set $\lambda = s$,

$$
L_\varphi = \frac{r^2 \sin^2\theta\dot{\varphi}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}
$$

Since the units of $\dot{\varphi} = \frac{d\varphi}{d\lambda}$ are of inverse length, L_φ has units of time, $\frac{m^2}{m\sqrt{m^2/sec^2}} = sec$.

We can find most of what we need from the two conserved quantities, E, L_φ, but we now digress to find the equations for the extremal time path and check the initial conditions.

3

1.3   Equations of the path

First, we reconsider the initial conditions. We know that the speed, $\frac{ds}{dt}$, and velocity components start at zero, but now we need to know the initial values of $\left[\left(\frac{dr}{ds}\right), \left(\frac{d\theta}{ds}\right), \left(\frac{d\varphi}{ds}\right)\right]$. These are related to the velocity by

$$\frac{dr}{ds} = \frac{dt}{ds}\frac{dr}{dt}$$
$$\frac{d\theta}{ds} = \frac{dt}{ds}\frac{d\theta}{dt}$$
$$\frac{d\varphi}{ds} = \frac{dt}{ds}\frac{d\varphi}{dt}$$

and each of these becomes ambiguous since $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}}$ diverges. We can get help from the constants of motion. From the energy relation,

$$v = \frac{ds}{dt} = \sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}$$

we have

$$\frac{dr}{ds} = \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{dr}{dt}$$

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d\theta}{dt}$$

$$\frac{d\varphi}{ds} = \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d\varphi}{dt}$$

Then, since L_φ must be non-zero and finite, its expression at the initial conditions gives

$$L_\varphi = \frac{r^2 \sin^2\theta}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\dot{\varphi}$$

$$= \frac{R^2 \sin^2\frac{\pi}{2}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\left(\frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d\varphi}{dt}\right)$$

$$= \frac{R^2}{Rg\left(1 - \frac{r^2}{R^2}\right)}\frac{d\varphi}{dt}$$

which means that

$$\frac{d\varphi}{dt} = \frac{L_\varphi g}{R}\left(1 - \frac{r^2}{R^2}\right)$$

and the initial behavior of $\dot{\varphi}$ must be

$$\frac{d\varphi}{ds} = \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d\varphi}{dt}$$

$$= \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{L_\varphi g}{R}\left(1 - \frac{r^2}{R^2}\right)$$

$$= \frac{1}{\sqrt{Rg}}\frac{L_\varphi g}{R}\sqrt{\left(1 - \frac{r^2}{R^2}\right)}$$

4

which also vanishes at $r = R$. For θ and for r we must use the equations for the path.

We vary the time functional with respect to each coordinate function. Since we must vary before setting $\lambda = s$, it simplifies the notation to notice that

$$
ds = \sqrt{\left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\theta}{d\lambda}\right)^2 + r^2 \sin^2\theta \left(\frac{d\varphi}{d\lambda}\right)^2}\, d\lambda
$$

$$
\dot{s} \equiv \frac{ds}{d\lambda} = \sqrt{\left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\theta}{d\lambda}\right)^2 + r^2 \sin^2\theta \left(\frac{d\varphi}{d\lambda}\right)^2}
$$

so that when we set $\lambda = s$ we have

$$
1 = \frac{ds}{ds} = \sqrt{\left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 + r^2 \sin^2\theta \left(\frac{d\varphi}{ds}\right)^2}
$$

$$
1 = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2}
$$

1.3.1   Vary $r(\lambda)$

Varying the fixed curve, but holding the endpoints fixed, we have three conditions. For δr,

$$
0 = \delta_r t\left[r(\lambda), \theta(\lambda), \varphi(\lambda)\right]
$$

$$
= \int \left[\frac{1}{2}\frac{1}{\dot{s}}\frac{2\dot{r}\delta\dot{r} + 2r\delta r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} - \frac{1}{2}\frac{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2}}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}Rg\left(-\frac{2r\delta r}{R^2}\right)\right]d\lambda
$$

$$
= \int \left[-\frac{d}{d\lambda}\left(\frac{1}{\dot{s}}\frac{\dot{r}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\right) + \frac{1}{\dot{s}}\frac{r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} + \frac{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2}}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}\frac{gr}{R}\right]\delta r\, d\lambda
$$

so the first equation of the path is

$$
\frac{d}{d\lambda}\frac{1}{\dot{s}}\frac{\dot{r}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} = \frac{1}{\dot{s}}\frac{r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} + \frac{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2}}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}\frac{gr}{R}
$$

and we may now set $\lambda = s$ and therefore $\dot{s} = 1$. Notice also that

$$
r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) = \frac{1}{r}\left(r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)
$$

$$
= \frac{1}{r}\left(1 - \dot{r}^2\right)
$$

we substitute and carry out the differentiation,

$$
\frac{d}{ds}\frac{\dot{r}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} = \frac{1 - \dot{r}^2}{r\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} + \frac{gr}{R\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}
$$

$$
\frac{\ddot{r}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} - \frac{1}{2}\frac{\dot{r}}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}Rg\left(-\frac{2r\dot{r}}{R^2}\right) = \frac{1 - \dot{r}^2}{r\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} + \frac{gr}{R\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}
$$

$$
\ddot{r} + \frac{r\dot{r}^2}{R^2 - r^2} = \frac{1 - \dot{r}^2}{r} + \frac{r}{R^2 - r^2}
$$

5

so finally

$$\begin{aligned} \ddot{r} &= \frac{1 - \dot{r}^2}{r} + \frac{r\left(1 - \dot{r}^2\right)}{R^2 - r^2} \\ &= \frac{1}{r}\left(1 - \dot{r}^2\right)\left(1 + \frac{r^2}{R^2 - r^2}\right) \\ &= \frac{R^2}{r}\frac{1 - \dot{r}^2}{R^2 - r^2} \end{aligned}$$

The divergence of this expression is because the derivatives are with respect to s, not time:

$$\ddot{r} = \frac{R^2}{r}\frac{1 - \dot{r}^2}{R^2 - r^2}$$

where

$$\begin{aligned} \ddot{r} &= \frac{d\dot{r}}{ds} \\ &= \frac{dt}{ds}\frac{d\dot{r}}{dt} \\ &= \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d}{dt}\left(\frac{dt}{ds}\frac{dr}{dt}\right) \\ &= \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d}{dt}\left(\frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{dr}{dt}\right) \\ &= \frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\left(\frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d^2r}{dt^2} - \frac{1}{2}\frac{Rg\left(-\frac{2r}{R^2}\frac{dr}{dt}\right)}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}\frac{dr}{dt}\right) \\ &= \frac{1}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^2}\left(\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}\frac{d^2r}{dt^2} + \frac{rg}{R}\left(\frac{dr}{dt}\right)^2\right) \end{aligned}$$

With the initial conditions

$$\begin{aligned} \frac{dr}{dt} &= 0 \\ \frac{d^2r}{dt^2} &= g \end{aligned}$$

this becomes

$$\ddot{r}_0 = \lim_{r \longrightarrow R}\frac{g}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^{3/2}}$$

If we set $r = R - \varepsilon$, then

$$\begin{aligned} \ddot{r}_0 &= \lim_{r \longrightarrow R}\frac{g}{\left[\frac{g}{R}\left(2R\varepsilon - \varepsilon^2\right)\right]^{3/2}} \\ &= \lim_{r \longrightarrow R}\frac{g}{\left[2g\varepsilon\right]^{3/2}} \end{aligned}$$

we see that $\ddot{r}_0$ diverges towards the initial value.

1.3.2 Vary $\theta(\lambda)$:

Next,

$$
\begin{aligned}
0 &= \delta_\theta t \\
&= \int \frac{r^2\dot\theta\delta\dot\theta + r^2\sin\theta\cos\theta\dot\varphi^2\delta\theta}{\sqrt{\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\varphi^2}\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}d\lambda \\
&= \int \left[-\frac{d}{d\lambda}\frac{r^2\dot\theta}{\sqrt{\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\varphi^2}\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}} + \frac{r^2\sin\theta\cos\theta\dot\varphi^2}{\sqrt{\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\varphi^2}\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}\right]\delta\theta d\lambda
\end{aligned}
$$

so that with $\lambda = s$,

$$
\frac{d}{d\lambda}\left(\frac{r^2\dot\theta}{\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}\right) = \frac{r^2\sin\theta\cos\theta\dot\varphi^2}{\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}
$$

$$
\frac{r^2\ddot\theta + 2r\dot r\dot\theta}{\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}} - \frac{1}{2}\frac{r^2\dot\theta Rg\left(-\frac{2r}{R^2}\dot r\right)}{\left[Rg\left(1-\frac{r^2}{R^2}\right)\right]^{3/2}} = \frac{r^2\sin\theta\cos\theta\dot\varphi^2}{\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}
$$

$$
r^2\ddot\theta + 2r\dot r\dot\theta + \frac{r^3 g\dot r\dot\theta}{R^2 g\left(1-\frac{r^2}{R^2}\right)} = r^2\sin\theta\cos\theta\dot\varphi^2
$$

$$
\ddot\theta = \sin\theta\cos\theta\dot\varphi^2 - \frac{2}{r}\dot r\dot\theta - \frac{rg\dot r\dot\theta}{R^2 g\left(1-\frac{r^2}{R^2}\right)}
$$

Now examine the early acceleration,

$$
\ddot\theta = \sin\theta\cos\theta\dot\varphi^2 - \frac{2}{r}\dot r\dot\theta - \frac{rg\dot r\dot\theta}{R^2 g\left(1-\frac{r^2}{R^2}\right)}
$$

Substituting for r and $\dot r$ as above, using L_φ,

$$
\dot\varphi = \frac{L_\varphi\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}{r^2\sin^2\theta}
$$

and with

$$
\begin{aligned}
\dot\theta &= \frac{d\theta}{ds} \\
&= \frac{dt}{ds}\frac{d\theta}{dt} \\
&= \frac{1}{\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}\frac{d\theta}{dt} \\
\ddot\theta &= \frac{1}{\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}\frac{d}{dt}\frac{1}{\sqrt{Rg\left(1-\frac{r^2}{R^2}\right)}}\frac{d\theta}{dt} \\
&= \frac{1}{Rg\left(1-\frac{r^2}{R^2}\right)}\frac{d^2\theta}{dt^2} + \frac{g\left(\frac{r}{R}\right)}{\left[Rg\left(1-\frac{r^2}{R^2}\right)\right]^2}\frac{dr}{dt}\frac{d\theta}{dt}
\end{aligned}
$$

we have

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\varphi}^2 - \frac{2}{r}\dot{r}\dot{\theta} - \frac{r\dot{r}\dot{\theta}}{R^2\left(1 - \frac{r^2}{R^2}\right)}$$

$$\frac{1}{Rg\left(1 - \frac{r^2}{R^2}\right)}\frac{d^2\theta}{dt^2} + \frac{g\left(\frac{r}{R}\right)}{\left[Rg\left(1 - \frac{r^2}{R^2}\right)\right]^2}\frac{dr}{dt}\frac{d\theta}{dt} = \cos\frac{\pi}{2}\frac{L_\varphi^2 Rg\left(1 - \frac{r^2}{R^2}\right)}{r^4\sin^4\frac{\pi}{2}} - \frac{2}{R}\frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{dr}{dt}\frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d\theta}{dt} - \frac{1}{R\left(1 - \frac{r^2}{R^2}\right)}$$

$$\frac{1}{Rg\left(1 - \frac{r^2}{R^2}\right)}\frac{d^2\theta}{dt^2} = \left(-\frac{2}{R}\frac{1}{Rg\left(1 - \frac{r^2}{R^2}\right)} - \frac{1}{R^2g\left(1 - \frac{r^2}{R^2}\right)^2} - \frac{1}{R^2g\left(1 - \frac{r^2}{R^2}\right)^2}\right)\frac{dr}{dt}\frac{d\theta}{dt}$$

$$\frac{1}{Rg\left(1 - \frac{r^2}{R^2}\right)}\frac{d^2\theta}{dt^2} = \left[-\frac{2}{R^2g\left(1 - \frac{r^2}{R^2}\right)} - \frac{2}{R^2g\left(1 - \frac{r^2}{R^2}\right)^2}\right]\frac{dr}{dt}\frac{d\theta}{dt}$$

$$\frac{d^2\theta}{dt^2} = -\frac{2}{R}\left[\frac{2 - \frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}}\right]\frac{dr}{dt}\frac{d\theta}{dt}$$

Now, expand in time:

$$r = R - \frac{1}{2}at^2$$

$$\theta = \frac{\pi}{2} + \frac{1}{2}bt^2$$

Then

$$b = \frac{2}{R}\left[\frac{1 + \frac{at^2}{R}}{\frac{at^2}{R}}\right]abt^2$$

$$= \frac{2}{R}\left[\frac{R + at^2}{at^2}\right]abt^2$$

$$= \frac{2}{R}\left(R + at^2\right)b$$

$$\left(1 + \frac{2a}{R}t^2\right)b = 0$$

and we must have $b = 0$. Therefore, the initial $\ddot{\theta}$ also vanishes and $\theta = \frac{\pi}{2}$ throughout the motion.

1.3.3   Vary $\varphi(\lambda)$:

Finally,

$$0 = \delta_\varphi t$$

$$= \int\left[-\frac{d}{d\lambda}\frac{r^2\sin^2\theta\dot{\varphi}}{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2}\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\right]\delta\varphi d\lambda$$

so that

$$\frac{d}{d\lambda}\left(\frac{r^2\sin^2\theta\dot{\varphi}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\right) = 0$$

This gives our previous constant of the motion,

$$L_\varphi = \frac{r^2\sin^2\theta\dot{\varphi}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}$$

8

with the initial conditions for φ,

$$\dot{\varphi} = \frac{L_\varphi}{R^2}\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}$$

and with a time derivative,

$$
\begin{aligned}
\frac{1}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{d\varphi}{dt} &= \frac{L_\varphi}{R^2}\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)} \\
\frac{d\varphi}{dt} &= \frac{L_\varphi g}{R}\left(1 - \frac{r^2}{R^2}\right)
\end{aligned}
$$

which vanishes as required.

1.4 Using the constants of motion to find the path

We would like to solve for the path of motion and the total time, and we can start from the constants of motion. Setting the initial conditions, we have:

$$
\begin{aligned}
L_\varphi &= \frac{r^2\dot{\varphi}}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}} \\
v &= \sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}
\end{aligned}
$$

where $v = \frac{ds}{dt} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2}$.

1.4.1 Solving for the path

We wish to find $\frac{dr}{dt}$. The expression for v gives:

$$
\begin{aligned}
\left(\frac{ds}{dt}\right)^2 &= Rg\left(1 - \frac{r^2}{R^2}\right) \\
\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2 &= Rg\left(1 - \frac{r^2}{R^2}\right) \\
\left(\frac{dr}{dt}\right)^2 &= Rg\left(1 - \frac{r^2}{R^2}\right) - r^2\left(\frac{d\varphi}{dt}\right)^2
\end{aligned}
$$

Next, eliminate $\frac{d\varphi}{dt}$,

$$
\begin{aligned}
L_\varphi &= \frac{r^2}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\dot{\varphi} \\
&= \frac{r^2}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)}}\frac{dt}{ds}\frac{d\varphi}{dt} \\
\frac{d\varphi}{dt} &= \frac{L_\varphi}{r^2}\frac{ds}{dt}\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right)} \\
&= \frac{L_\varphi Rg}{r^2}\left(1 - \frac{r^2}{R^2}\right)
\end{aligned}
$$

9

Combining this with the previous expression,

$$\left(\frac{dr}{dt}\right)^2 = Rg\left(1 - \frac{r^2}{R^2}\right) - r^2\left(\frac{d\varphi}{dt}\right)^2$$

$$= Rg\left(1 - \frac{r^2}{R^2}\right) - r^2\left(\frac{L_\varphi Rg}{r^2}\left(1 - \frac{r^2}{R^2}\right)\right)^2$$

$$\frac{dr}{dt} = \sqrt{Rg\left(1 - \frac{r^2}{R^2}\right) - L_\varphi^2 g^2 \frac{R^2}{r^2}\left(1 - \frac{r^2}{R^2}\right)^2}$$

$$dt = \frac{dr}{\sqrt{Rg\left(1 - \frac{r^2}{R^2}\right) - \frac{L_\varphi^2 R^2 g^2}{r^2}\left(1 - \frac{r^2}{R^2}\right)^2}}$$

$$dt = \frac{r\,dr}{\sqrt{Rg\left(r^2 - \frac{r^4}{R^2}\right) - L_\varphi^2 R^2 g^2\left(1 - \frac{2r^2}{R^2} + \frac{r^4}{R^4}\right)}}$$

Now integrate, letting $\zeta = r^2$, and

$$\omega = \frac{1}{R}\sqrt{L_\varphi^2 g^2 + Rg}$$

$$\omega^2 R^2 - Rg = L_\varphi^2 g^2$$

$$t = \int \frac{r\,dr}{\sqrt{Rgr^2\left(1 - \frac{r^2}{R^2}\right) - L_\varphi^2 R^2 g^2\left(1 - \frac{2r^2}{R^2} + \frac{r^4}{R^4}\right)}}$$

$$= \frac{1}{2}\int \frac{d\zeta}{\sqrt{-L_\varphi^2 R^2 g^2 + \left(Rg + 2L_\varphi^2 g^2\right)\zeta - \frac{1}{R^2}\left(L_\varphi^2 g^2 + gR\right)\zeta^2}}$$

$$= \frac{1}{2}\int \frac{d\zeta}{\sqrt{-\omega^2 R^4 + R^3 g + \left(2\omega^2 R^2 - Rg\right)\zeta - \omega^2\zeta^2}}$$

The integral of this is similar to things we have done, and should yield the hypocycloid. Complete the square,

$$-\omega^2 R^4 + R^3 g + \left(2\omega^2 R^2 - Rg\right)\zeta - \omega^2\zeta^2 = -\omega^2\zeta^2 + \left(2\omega^2 R^2 - Rg\right)\zeta - \omega^2 R^4 + R^3 g$$

$$= -\left(\omega^2\zeta^2 - \left(2\omega^2 R^2 - Rg\right)\zeta\right) - \omega^2 R^4 + R^3 g$$

$$= -\left(\omega^2\zeta^2 - \left(2\omega^2 R^2 - Rg\right)\zeta + \frac{1}{4\omega^2}\left(2\omega^2 R^2 - Rg\right)^2\right) - \omega^2 R^4 + R^3 g + \frac{1}{4\omega^2}\left(2\omega^2 R^2 - Rg\right)^2$$

$$= -\left(\omega\zeta - \frac{1}{2\omega}\left(2\omega^2 R^2 - Rg\right)\right)^2 - \omega^2 R^4 + R^3 g + \frac{1}{4\omega^2}\left(4\omega^4 R^4 - 4\omega^2 R^3 g + R^2 g^2\right)$$

$$= -\left(\omega\zeta - \frac{1}{2\omega}\left(2\omega^2 R^2 - Rg\right)\right)^2 + \frac{R^2 g^2}{4\omega^2}$$

Now let $A = \frac{1}{2\omega}\left(2\omega^2 R^2 - Rg\right)$ and $B = \frac{Rg}{2\omega}$ so that

$$t = \frac{1}{2}\int \frac{d\zeta}{\sqrt{-\left(\omega\zeta - A\right)^2 + B^2}}$$

$$t = \frac{1}{2\omega}\int \frac{d\xi}{\sqrt{B^2 - \xi^2}}$$

10

so that letting $\xi = \omega\zeta - A = B\sin\alpha$

$$
\begin{aligned}
t &= \frac{\alpha - \alpha_0}{2\omega} \\
\alpha &= (2\omega t + \alpha_0) \\
\frac{\omega r^2 - A}{B} &= \sin(2\omega t + \alpha_0) \\
r^2 &= \frac{1}{\omega}(A + B\sin(2\omega t + \alpha_0))
\end{aligned}
$$

We show below that this describes a hypocycloid, once we use the formula to find the time of a cycle.

1.4.2   Time for a circuit

This expression for r^2 is periodic, and must be maximum at $t = 0$, with $r(t=0) = R$. This means that we need to choose $\alpha_0 = \frac{\pi}{2}$,

$$
\begin{aligned}
r^2 &= \frac{A}{\omega} + \frac{B}{\omega}\cos(2\omega t) \\
R^2 &= \frac{A+B}{\omega}
\end{aligned}
$$

$$
\begin{aligned}
r^2 &= \frac{A}{\omega} + \frac{B}{\omega}\cos(2\omega t) \\
R^2 &= \frac{A+B}{\omega} \\
&= \frac{\frac{1}{2\omega}(2\omega^2 R^2 - Rg) + \frac{Rg}{2\omega}}{\omega} \\
&= \frac{2\omega^2 R^2}{2\omega^2} \\
&= R^2
\end{aligned}
$$

The solution returns to a maximum in a time $2\omega t = 2\pi$,

$$
\begin{aligned}
t &= \frac{\pi}{\omega} \\
&= \frac{\pi R}{\sqrt{L_\varphi^2 g^2 + Rg}}
\end{aligned}
$$

Our solution is therefore:
$$
r^2 = R^2 - \frac{Rg}{2\omega^2}(1 - \cos(2\omega t))
$$

From this we see that the maximum depth below Earth's surface is $\frac{Rg}{\omega^2}$.

1.4.3   Equation for a hypocycloid

Now we show that the path describes a hypocycloid. From Wolfram, we have the Cartesian components,

$$
\begin{aligned}
x &= (a-b)\cos\phi + b\cos\frac{a-b}{b}\phi \\
y &= (a-b)\sin\phi + b\sin\frac{a-b}{b}\phi
\end{aligned}
$$

11

where the larger radius is a and the smaller is b. Changing to polar coordinates,

$$
\begin{aligned}
r^2 &= \left[(a-b)\cos\phi + b\cos\frac{a-b}{b}\phi\right]^2 + \left[(a-b)\sin\phi + b\sin\frac{a-b}{b}\phi\right]^2 \\
&= (a-b)^2\cos^2\phi + 2(a-b)b\cos\phi\cos\frac{a-b}{b}\phi + b^2\cos^2\frac{a-b}{b}\phi \\
&\quad + (a-b)^2\sin^2\phi + 2(a-b)b\sin\phi\sin\frac{a-b}{b}\phi + b^2\sin^2\frac{a-b}{b}\phi \\
&= (a-b)^2 + b^2 + 2(a-b)b\left(\cos\phi\cos\frac{a-b}{b}\phi + \sin\phi\sin\frac{a-b}{b}\phi\right) \\
&= \left[(a-b)^2 + b^2\right] + 2(a-b)b\cos\left(\phi - \frac{a-b}{b}\phi\right) \\
&= \left[(a-b)^2 + b^2\right] + 2(a-b)b\cos\left(\frac{2b-a}{b}\phi\right)
\end{aligned}
$$

and equating,

$$
\left[(a-b)^2 + b^2\right] + 2(a-b)b\cos\left(\frac{a-2b}{b}\phi\right) = \frac{1}{\omega}(A + B\cos(2\omega t))
$$

and comparing coefficients, we have

$$
\begin{aligned}
\frac{A}{\omega} &= (a-b)^2 + b^2 \\
\frac{B}{\omega} &= 2(a-b)b \\
2\omega t &= \frac{a-2b}{b}\phi
\end{aligned}
$$

Adding A and B,

$$
\begin{aligned}
R^2 &= \frac{A+B}{\omega} \\
&= (a-b)^2 + b^2 + 2(a-b)b \\
&= ((a-b)+b)^2 \\
&= a^2
\end{aligned}
$$

so the larger radius of the hypocycloid is $a = R$, the radius of Earth. Subtracting,

$$
\begin{aligned}
\frac{A-B}{\omega} &= \frac{\frac{1}{2\omega}\left(2\omega^2 R^2 - Rg\right) - \frac{Rg}{2\omega}}{\omega} \\
&= R^2 - \frac{Rg}{\omega^2} \\
R^2 - \frac{Rg}{\omega^2} &= (a-b)^2 - 2(a-b)b + b^2 \\
&= ((a-b)-b)^2 \\
&= (R-2b)^2 \\
R^2 - \frac{Rg}{\omega^2} &= R^2 - 4bR + 4b^2 \\
4b^2 - 4bR + \frac{Rg}{\omega^2} &= 0 \\
b &= \frac{4R \pm \sqrt{16R^2 - 16\frac{Rg}{\omega^2}}}{8}
\end{aligned}
$$

$$b = \frac{R - R\sqrt{1 - \frac{g}{\omega^2 R}}}{2}$$

$$b = \frac{1}{2}R\left(1 - \sqrt{\frac{L_\varphi^2 g^2}{L_\varphi^2 g^2 + Rg}}\right)$$

where we choose the minus sign to get the smallest value of b. This fixes the hypocycloid radii:

$$a = R$$

$$b = \frac{1}{2}R\left(1 - \sqrt{\frac{L_\varphi^2 g^2}{L_\varphi^2 g^2 + Rg}}\right)$$

$$r^2 = R^2 - \frac{Rg}{2\omega^2}\left(1 - \cos\left(2\omega t\right)\right)$$

and the equation of the hypocycloid. Fixing the second endpoint of the motion will determine the remaing constant, L_φ.

1.4.4 This section still in progress

To find the final constant of motion, we use the radius, b, of the smaller circle. The particle will reach the surface again when this circle has gone through one complete revolution. This happens when the small circle has rolled a distance equal to its circumference, $l = 2\pi b$. Let l be the distance from New York to San Francisco. Then we require L_φ such that

$$l = 2\pi b$$

$$= 2\pi \frac{1}{2}R\left(1 - \sqrt{\frac{L_\varphi^2 g^2}{L_\varphi^2 g^2 + Rg}}\right)$$

$$\frac{l}{\pi R} = 1 - \sqrt{\frac{L_\varphi^2 g^2}{L_\varphi^2 g^2 + Rg}}$$

$$\sqrt{\frac{L_\varphi^2 g^2}{L_\varphi^2 g^2 + Rg}} = 1 - \frac{l}{\pi R}$$

$$\frac{L_\varphi^2 g^2}{L_\varphi^2 g^2 + Rg} = \left(1 - \frac{l}{\pi R}\right)^2$$

$$L_\varphi^2 g^2 = \left(1 - \frac{l}{\pi R}\right)^2 L_\varphi^2 g^2 + \left(1 - \frac{l}{\pi R}\right)^2 Rg$$

$$\left[1 - \left(1 - \frac{2l}{\pi R} + \frac{l^2}{\pi^2 R^2}\right)\right] L_\varphi^2 g^2 = \left(1 - \frac{l}{\pi R}\right)^2 Rg$$

$$L_\varphi^2 g^2 = \frac{\left(1 - \frac{l}{\pi R}\right)^2 Rg}{\frac{2l}{\pi R}\left(1 - \frac{l}{2\pi R}\right)}$$

$$L_\varphi^2 = \frac{\pi R^2 \left(1 - \frac{l}{\pi R}\right)^2}{2lg\left(1 - \frac{l}{2\pi R}\right)}$$

$$L_\varphi = \sqrt{\frac{\pi}{2lg}} \frac{R\left(1 - \frac{l}{\pi R}\right)}{\sqrt{1 - \frac{l}{2\pi R}}}$$

This determines the frequency, ω, to be

$$\omega = \frac{1}{R}\sqrt{L_\varphi^2 g^2 + Rg}$$

$$= \sqrt{\frac{\pi g}{2l} \frac{\left(1 - \frac{l}{\pi R}\right)^2}{\left(1 - \frac{l}{2\pi R}\right)} + \frac{g}{R}}$$