Gauging Newton's Law

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Abstract

We derive both Lagrangian and Hamiltonian mechanics as gauge theories of Newtonian mechanics. Systematic development of the distinct symmetries of dynamics and measurement suggest that gauge theory may be motivated as a reconciliation of dynamics with measurement. Applying this principle to Newton's law with the simplest measurement theory leads to Lagrangian mechanics, while use of conformal measurement theory leads to Hamiltonian mechanics.

1 Introduction

Recent progress in field theory, when applied to classical physics, reveals a previously unknown unity between various treatments of mechanics. Historically, Newtonian mechanics, Lagrangian mechanics and Hamiltonian mechanics evolved as distinct formulations of the content of Newton's second law. Here we show that Lagrangian and Hamiltonian mechanics both arise as local gauge theories of Newton's second law.

While this might be expected of Lagrangian mechanics, which is, after all, just the locally coordinate invariant version of Newton's law, achieving Hamiltonian mechanics as a gauge theory is somewhat surprising. The reason it happens has to do with a new method of gauging scale invariance called *biconformal gauging.* The study of biconformal gauging of Newtonian mechanics serves a dual purpose. First, it sheds light on the meaning in field theory of biconformal gauging, which has already been shown to have symplectic structure and to lead to a satisfactory relativistic gravity theory. Second, we are now able to see a conceptually satisfying unification of Hamiltonian mechanics with its predecessors.

Beyond these reasons for the study, we find a hint of something deeper. Not only do many of the mathematical properties of Hamiltonian dynamics emerge necessarily, but also we are offered a tantalizing glimpse of a new possibility – this 6-dimensional space appears to be the proper arena for both classical and quantum physics. While the results presented here are purely classical, we revisit this possibility in our conclusion. A full discussion of biconformal spaces and quantum mechanics is given in [2].

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Although the present article makes only minor use of relativistic biconformal spaces, we give a brief account of their history and properties. The story starts with conformal gauge theories, which are notable for certain pathologies: (1) the requirement for an invariant action in $2n$ dimensions to be of n^{th} order in the curvature and/or the requirement for auxiliary fields to write a linear action, and (2) the presence of unphysical size changes. The existence of this alternative way to gauge the conformal group was first explored by Ivanov and Niederle [1], who were led to an eight dimensional manifold by gauging the conformal group of a four dimensional spacetime. They restricted the dependence on the extra four dimensions to the minimum needed for consistency with conformal symmetry. Later, Wheeler [3], generalizing to arbitrary dimensions, n, defined the class of biconformal spaces as the result of the $2n$-dim gauging without imposing constraints, showing it to have symplectic structure and admit torsion free spaces consistent with general relativity and electromagnetism. Wehner and Wheeler [4] went on to write the most general class of actions linear in the biconformal curvatures, eliminating problems (1) and (2) above, and showing that the resulting field equations lead to the Einstein field equations. Unlike previous conformal gauge theories, this action takes the same form in any dimension.

In the next two sections, we make some observations regarding dynamical laws, measurement theory and symmetry, then describe the global $ISO(3)$ symmetry of Newton's second law and the global $SO(4,1)$ symmetry of Newtonian measurement theory. In Sec. 4, we give two ways to make these different dynamical and measurement symmetries agree. After briefly describing our method of gauging in Sec. 5, we turn to the actual gauging of Newtonian mechanics. In Sec. 6 we show that the $ISO(3)$ gauge theory leads, as expected, to Lagrangian mechanics. This illustrates our method of gauging in a familiar context. Then, in Sec. 7 we show that biconformal gauging of the $SO\,(4,1)$ symmetry leads to Hamiltonian dynamics, including a discussion of multiple particles. In the penultimate section, we discuss an important question of interpretation, checking that there are no unphysical size changes. Finally, we end with a brief summary and some observations about the relationship between biconformal spaces and quantum physics.

2 What constitutes a physical theory?

The relationship of symmetry to the form of physical laws has a long history, including Galilean relativity, the extended discussion surrounding the transition from Newtonian to relativistic dynamics, and the elegant theorem of Noether. Many of these ideas are synthesized by Anderson [5], who makes careful distinctions between kinematical and dynamical trajectories, their covariance and symmetry groups, and the measurements that confirm them.

A number of the ideas discussed by Anderson concern us here, but with a slightly different emphasis. For example, Anderson discusses the class of "kinematically possible trajectories" whereas we refer below to the "physical arena". Clearly, these are closely related ideas, since a subset of paths in the arena constitutes the set of possible trajectories. As in [5], we find that symmetry requirements place a strong restriction on that class or arena. Indeed, we go a step further and use group theoretic methods to derive the arena from experimentally determined symmetries.

This brings us to another difference of emphasis, namely, our focus on the class of experimentally determined symmetries. This class is determined by what we call measurement theory. Quoting [5]:

> Every physical theory attempts to associate mathematical quantities of some kind with the elements of the physical system the theory is supposed to describe. How one makes this association is one of the most difficult parts of physics. . . .

Measurement theory is the set of rules we use to accomplish the association between mathematical quantities present in a dynamical theory and numbers resulting from experiments. We make important use of the distinction between dynamical symmetries and symmetries implicit in these rules of measurement. Understanding the role played by each of these will lead us to a deeper understanding of symmetry and gauge theory, and ultimately brings us back to questions about the arena for physical theory.

To clarify the difference between dynamical symmetry and measurement symmetry, we consider the distinction in three examples: (1) quantum mechanics, (2) classical mechanics, and (3) special relativity.

First, consider quantum theory where the dynamics and measurement theories are quite distinct from one another. The dynamical law of quantum mechanics is the Schrödinger equation,

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This equation gives the time evolution of a state, ψ, but the state has no direct physical meaning – we require a measurement theory. For this purpose we require a norm or an inner product on states,

$$\langle\psi \mid \psi\rangle = \int_V \psi^*\psi \; d^3x$$

to give a measurable number. In addition, auxiliary rules for interpretation are needed. For example, the quantum norm above is interpreted as the probability of finding the particle characterized by the state ψ in the volume V. Additional rules govern measurement of the full range of dynamical variables.

For our second example, we identify these same elements of Newtonian mechanics. Newtonian mechanics is so closely tied to our intuitions about how things move that we don't usually separate dynamics and measurement as conceptually distinct. Still, now that we know what we are looking for it is not difficult. The dynamical law, of course, is Newton's second law:

$$F^i = m\frac{dv^i}{dt}$$

which describes the time evolution of a position vector of a particle. The measurement theory goes back to the Pythagorean theorem – it is based on the line element or vector length in

Euclidean space:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= \eta_{ij} dx^i dx^j \\ \mathbf{v} \cdot \mathbf{w} &= \eta_{ij} v^i w^j \end{aligned}$$

where

$$\eta_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

is the Euclidean metric in Cartesian coordinates. It is metric structure that provides measurable numbers from the position vectors, forces and other elements related by the dynamical equation. As we shall see below, there are also further rules required to associate quantities computed from the dynamical laws with numbers measured in the laboratory.

Finally, the separation between dynamical law and measurement theory in special relativity is quite similar to that in classical mechanics. The law of motion is just the four dimensional version of Newton's second law, while the measurement theory is now based on the Minkowski line element and inner product,

$$\begin{aligned} ds^2 &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= \eta_{\alpha\beta} dx^\alpha dx^\beta \\ \mathbf{v} \cdot \mathbf{w} &= \eta_{\alpha\beta} v^\alpha w^\beta \end{aligned}$$

where α and β now run from 0 to 3.

Once we have both a dynamical law and a measurement theory, we can begin detailed exploration of the physical theory. Generally, this means analyzing the nature of different interactions and making predictions about the outcomes of experiments. For these two purposes – studying interactions and making predictions – the most important tool is symmetry. The use of symmetry for studying interactions follows from the techniques of gauge theory, in which a dynamical law with a global symmetry is modified to be consistent with a local symmetry of the same type. This procedure introduces new fields into the theory, and these new fields generally describe interactions. The use of symmetry for prediction relies on Noether's theorem, which guarantees a conserved quantity corresponding to any continuous symmetry. Once we have such a conserved quantity, we have an immediate prediction: the conserved quantity will have the same value in the future that it has now.

These three properties – dynamics, measurement, and symmetry – play a role in every meaningful physical theory. We now revisit our three examples to look at the available symmetries.

First, we note that in quantum theory both the dynamical law and the measurement theory are invariant under certain multiples of the wave function. The dynamical law is linear, hence consistent with arbitrary multiples of solutions. However, because of the derivatives involved in the Schrödinger equation, these multiples must be global, $\psi \to A_0 e^{i\varphi_0}\psi$. In contrast to this, the quantum norm is preserved by local multiples only if the multiple is a pure phase:

$$\psi \rightarrow e^{i\varphi(x)}\psi$$

Thus, the dynamical law and the measurement theory have different symmetries. Of course, $U(1)$ gauge theory and the usual normalization of the wave function provide one means of reconciling this difference. The reconciliation involves two ways of modifying the symmetry of the dynamical equation to agree with that of the measurement theory – first by restriction (fixing A_0 to normalize ψ) and second by extension (modifying the dynamical law to be consistent with local $U(1)$ transformations).

Gauging the $U(1)$ phase symmetry plays an extremely important role. By the general procedure of gauging, we replace global symmetries by local ones, and at the same time replace the dynamical law by one consistent with the enlarged symmetry. Well-defined techniques are available for accomplishing the required change in the dynamical laws. When the gauging procedure is applied to the phase invariance of quantum field theory, the result is a theory that includes electromagnetism. Thus, the gauging procedure provides a way to systematically introduce interactions between particles, i.e., forces.

In our second example, the symmetry of Newtonian mechanics is often taken to be the set of transformations relating inertial frames. We can arrive at this conclusion by asking what transformations leave the dynamical equation invariant. The answer is that Newton's second law is invariant under any transformation of the form

$$\begin{aligned} x^i &\rightarrow J^i{}_j x^j + v^i t + x_0^i \\ F^i &\rightarrow J^i{}_j F^j \end{aligned}$$

where $J^i{}_j$ is a constant, nondegenerate matrix and v^i and x_0^i are constant vectors. A shift in the time coordinate and time reversal are also allowed. If we restrict $J^i{}_j$ to be orthogonal these comprise the Galilean transformations. However, not all of these are consistent with Newtonian measurement theory. If we ask which of the transformations above also preserve the Pythagorean norm, we must further restrict the transformation of x^i to be homogeneous. The combined measurement and dynamical theories are thus invariant under

$$\begin{aligned} x^i &\rightarrow O^i{}_j x^j \\ F^i &\rightarrow O^i{}_j F^j \\ t &\rightarrow t + t_0 \end{aligned}$$

While this brief argument leads us to the set of orthogonal inertial frames, it is not systematic. Rather, as we shall see, this is a conservative estimate of the symmetries that are possible. In particular, the infinitesimal line element is invariant under general coordinate transformations.

Finally, the relativistic version of Newton's second law transforms covariantly under global Lorentz transformations and global translations. In the measurement theory however, the line element is invariant under general coordinate transformations. Reconciling this difference by gauging, thereby making the dynamical laws invariant under local Lorentz transformations, provides a successful theory of gravity – general relativity. As we shall see, gauging works well in the Newtonian case too. Although we will not look for new interactions in the Newtonian gauge theory (such as Euclidean gravity), we will see that gauging leads directly to both the Lagrangian and Hamiltonian formulations of mechanics.

In the next sections, we treat the symmetries of Newtonian mechanics in a more systematic way. In preparation for this, recall that in the quantum phase example, we both restricted and extended the dynamical law to accommodate a symmetry of the measurement theory, but arriving at the inertial frames for the Newtonian example we only restricted the symmetry of the dynamical law. This raises a general question. When the dynamical law and measurement theory have different symmetries, what do we take as the symmetry of the theory? Clearly, we should demand that the dynamical equations and the measurement theory share a common set of symmetry transformations. If there is a mismatch, we have three choices:

1. Restrict the symmetry to a subset shared by both the dynamical laws and the measurement theory.

2. Generalize the measurement theory to one with the same symmetry as the dynamical law.

3. Generalize the dynamical equation to one with the same symmetry as the measurement theory.

We annunciate and apply the *Goldilocks Principle:* Since we recognize that symmetry sometimes plays an important predictive role in specifying possible interactions, option #1 is *too small.* It is unduly restrictive, and we may miss important physical content. By contrast, the symmetry of measurement is *too large* for option #2 to work – inner products generally admit a larger number of symmetries than dynamical equations. Option #3 is *just right*: there are general techniques for enlarging the symmetry of a dynamical equation to match that of a measurement theory. Indeed, this is precisely what happens in gauge theories. The extraordinary success of gauge theories may be because they extend the dynamical laws to agree with the maximal information permitted within a given measurement theory.

We will take the point of view that the largest possible symmetry is desirable, and will therefore always try to write the dynamical law in a way that respects the symmetry of our measurement theory. This leads to two novel gaugings of Newton's law. In the next section we look in detail at two symmetries of the second law: the usual Euclidean symmetry, $ISO(3)$, and the $SO(4,1)$ conformal symmetry of a modified version of Newton's law. Each of these symmetries leads to an interesting gauge theory.

3 Two symmetries of classical mechanics

In this section we first find the symmetry of Newton's second law, then find the symmetry of Newtonian measurement theory. We conclude the section with some observations on the nature of these symmetries and the relationship between them.

3.1 Symmetry of the dynamical equation

Newton's second law

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} \tag{1}$$

has several well-known symmetries [5]. For completeness, the point symmetries leaving eq.(1) invariant are derived in Appendix 1. The result is that two allowed coordinate systems must be related by a constant, inhomogeneous, general linear transformation, together with a shift (and possible time reversal) of t :

$$\begin{aligned} \tilde{x}^m &= J^m_{\ n}x^n + v_0^m t + x_0^m && (2) \\ \tilde{t} &= t + t_0 && (3) \\ \tilde{F}^m &= J^m_{\ n}F^n && (4) \end{aligned}$$

where $J^m_{\ n}$ is any constant, non-degenerate matrix, v_0^m and x_0^m are arbitrary constant vectors, and t_0 is any real constant. Notice that, setting $e^\lambda = |\det(J^m_{\ n})|$, Newton's second law transforms covariantly with respect to global rescaling of units, $x^m \to e^\lambda x^m$. We can consider scalings of the other quantities (F^i and t) as well.

Eqs.(2-4) gives a 16-parameter family of transformations: nine for the independent components of the 3×3 matrix J, three for the boosts v_0^i, three more for the arbitrary translation, x_0^m, and a single time translation. The collection of all of these coordinate sets constitutes the maximal set of inertial systems. This gives us the symmetry of the dynamical law.

3.2 Symmetry of Newtonian measurement theory

Newtonian measurement theory begins with the Pythagorean theorem as embodied in the line element and corresponding vector product

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 = \eta_{ij}dx^i dx^j && (5) \\ \mathbf{v}\cdot\mathbf{w} &= \eta_{ij}v^i w^j && (6) \end{aligned}$$

The line element is integrated to find lengths of curves, while the dot product lets us find components of vectors by projecting on a set of basis vectors. The symmetry preserving these is $SO(3)$, the invariance group of the Euclidean metric η_{ij}. Without introducing a connection, this group must be global to preserve the Euclidean vector space . However, the infinitesimal line element is preserved by general coordinate transformations, equivalent to invariance under local rotations and translations – the local inhomogeneous orthogonal group, $ISO(3)$. It is this difference between global and local invariance that is addressed by gauge theory.

Regardless of whether the symmetry is local or global, a line element or an inner product is not a complete theory of measurement. We must be specific about how the numbers found from the inner product relate to numbers measured in the laboratory.

Suppose we wish to characterize the magnitude of a displacement vector, $d\mathbf{x}$, separating two particles. Using the Euclidean line element,

$$ds^2 = \eta_{ij} dx^i dx^j \tag{7}$$

The result, to be meaningful, must still be expressed in some set of units, say, meters or centimeters. The fact that either meters or centimeters will do may be expressed by saying that we work with an equivalence class of metrics differing by a positive multiplier. Thus, if we write length ds in meters as ds_m, then to give the length ds_{cm} in centimeters we write

$$ds_{cm} = 10^2 ds_m$$

A quantity which is invariant under such changes of units is the ratio of any two lengths,

$$\frac{ds_{1m}}{ds_{2m}} = \frac{ds_{1cm}}{ds_{2cm}}$$

Transformations which leave such ratios invariant produce no measurable physical effect. Associating the scale factor with the metric, we regard all metrics of the form

$$g_{ij} = e^{2\lambda} \eta_{ij}$$

as equivalent. The factor $e^{2\lambda}$ is called a conformal factor; two metrics which differ by a conformal factor are conformally equivalent.

The symmetry group which preserves conformal equivalence classes of metrics is the conformal group, locally isomorphic to $SO(4,1)$. The (global) conformal group is comprised of the following transformations:

$$y^i = \begin{cases} O^i{}_j x^j & \textit{Orthogonal transformation} \\ x^i + a^i & \textit{Translation} \\ e^{\lambda} x^i & \textit{Dilatation} \\ \frac{x^i + x^2 b^i}{1 + 2b \cdot x + b^2 x^2} & \textit{Special conformal transformation} \end{cases}$$

The first three of these are familiar symmetries. We now discuss each of the conformal symmetries, and the relationship between the $SO(4,1)$ symmetry of classical measurement theory and the $ISO(3)$ symmetry of the dynamical law.

3.3 Relationship between the dynamical and measurement symmetries

Newton's second law, the dynamical equation of classical mechanics, is invariant under global changes of inertial frame. Newtonian measurement theory, by contrast, is invariant under the corresponding conformal group, $SO(4,1)$. For the invariance of ratios of infinitesimal line elements the symmetry may be local. Before seeking agreement between these different symmetries, we consider the relationship between the inertial transformations and global $SO(4,1)$. We also introduce some nomenclature relevant to dilatations and special conformal transformations.

3.3.1 Orthogonal transformations and translations

As expected, there are some simple relationships between the symmetries of Newton's second law and the conformal symmetries of the Euclidean line element. Of the global conformal transformations, the first three – orthogonal transformations, translations, and dilatations – all are allowed transformations to new inertial frames. We only need to restrict the global general linear transformations $J^m_{\ n}$ of eqs.(2) and (4) to orthogonal, $O^m_{\ n}$ for these to agree, while the $v_0^m t + x_0^m$ part of eq.(2) is a parameterized global translation.

3.3.2 Dilatations

For dilatations we see the invariance of Newton's second law simply because the units on both sides of the equation match:

$$\begin{aligned} [\mathbf{F}] &= \frac{kg \cdot m}{s^2} \\ [m\mathbf{a}] &= kg \cdot \frac{m}{s^2} \end{aligned}$$

No matter how we scale mass, length and time, Newton's law is preserved. Notice that the conformal transformation of units considered here is completely different from the conformal transformations (or renormalization group transformations) often used in quantum field theory. The present transformations are applied to *all* dimensionful fields, and it is impossible to imagine this simple symmetry broken. By contrast, in quantum field theory only certain parameters are renormalized and there is no necessity for dilatation invariance.

To keep track of dilatations, it is useful to choose a uniform way to specify how quantities scale. Classical mechanics does not have any natural fundamental constants which could be used to convert mass and time units into units of length as we might expect from a totally geometric theory. Such constants *do* exist in relativity (c converts time to a length) and quantum mechanics ($\frac{1}{\hbar c}$ converts mass to $(length)^{-1}$). Nonetheless, as we shall see, gauging the conformal group expresses phase space variables in geometric units, and it is therefore useful to think of all units as powers of length. To do this without using c or $\hbar$ is simply a matter of choosing an arbitrary velocity, say $v_0 = 1\frac{m}{\mathrm{sec}}$, and an arbitrary unit of inverse action, for example $\alpha_0 = 1\frac{\mathrm{sec}}{kg \cdot m^2}$. Using these, all MKS units are easily rendered as lengths. The arbitrary constants v_0 and α_0 drop out of any physical prediction. Notice, however, that the existence of $\hbar$ and c in relativistic quantum theories suggests that a relativistic quantum theory in biconformal space could be naturally geometric.

Given an arbitrary dynamical variable A with MKS units

$$[A] = m^{\alpha} (kg)^{\beta} (\mathrm{sec})^{\gamma}$$

we immediately have

$$\left[A (\alpha_0 v_0)^{\beta} v_0^{\gamma}\right] = m^{\alpha - \beta + \gamma}$$

Then with the units of A expressed as $(length)^k$ with $k = \alpha - \beta + \gamma$, we immediately know that under a dilatation of the metric by a factor $e^{2\lambda}$, A will change according to

$$A \to e^{k\lambda} A$$

The number k is called the *conformal weight* of A. For example, force has weight $k = -2$ since we may write

$$\left[\frac{\alpha_0}{v_0}\mathbf{F}\right] = \frac{1}{l^2}$$

The norm of this vector then transforms as

$$\left\|\frac{\alpha_0}{v_0}\mathbf{F}\right\| \to e^{-2\lambda}\left\|\frac{\alpha_0}{v_0}\mathbf{F}\right\|$$

With this understanding, we see that Newton's law, eq.(1), transforms covariantly under global dilatations. With force as above and

$$\begin{aligned} [\alpha_0 m\mathbf{v}] &= \frac{1}{l} \\ \left[\frac{1}{v_0}\frac{d}{dt}\right] &= \frac{1}{l} \end{aligned}$$

the second law has units $(length)^{-2}$ throughout:

$$\left[\left(\frac{\alpha_0}{v_0}\right)\mathbf{F}\right] = \left[\left(\frac{1}{v_0}\frac{d}{dt}\right)(\alpha_0 m\mathbf{v})\right] = \frac{1}{l^2} \tag{8}$$

Under a global dilatation, we therefore have

$$e^{-2\lambda}\left(\frac{\alpha_0}{v_0}\right)\mathbf{F} = e^{-\lambda}\frac{1}{v_0}\frac{d}{dt}\left(e^{-\lambda}\alpha_0 m\mathbf{v}\right) = e^{-2\lambda}\frac{1}{v_0}\frac{d}{dt}(\alpha_0 m\mathbf{v}) \tag{9}$$

Newton's law is therefore globally dilatation covariant, of conformal weight -2. Notice that the arbitrary constants cancel.

3.3.3 Special conformal transformations

The story is very different for special conformal transformations. These surprising looking transformations are translations in inverse coordinates. Let a single "point at infinity", ω, provide a one point compactification of R^3. Such an added point has no measurable consequence since the time required to reach it at any finite velocity is infinite, and any information coming from such a point requires an infinite amount of time to reach us. Then we may define the unique inverse to any coordinate x^i as

$$y^i = -\frac{x^i}{x^2}$$

where the origin and ω are inverse to one another (see Appendix 2). Note that inversion is a discrete conformal transformation since

$$dy^i dy_i = \left(\frac{1}{x^2}\right)^2 dx^i dx_i$$

Sandwiching a translation between two inversions is therefore also conformal, and gives the general form of a special conformal transformation:

$$x^i \to -\frac{x^i}{x^2} \to -\frac{x^i}{x^2} - b^i \to \frac{x^i + x^2 b^i}{1 + 2b^i x_i + b^2 x^2}$$

The effect of a special conformal transformation on the line element is now easy to compute. Letting y^i be inverse to x^i and setting

$$\begin{aligned} w^i &= y^i + b^i \\ z^i &= -\frac{w^i}{w^2} \end{aligned}$$

the transformation $x^i \to z^i$ is a special conformal transformation. Then we have

$$\begin{aligned} \eta_{ij} dz^i dz^j &= \left(\frac{1}{w^2}\right)^2 \eta_{ij} dw^i dw^j \\ &= \left(\frac{1}{w^2}\right)^2 \eta_{ij} dy^i dy^j \\ &= \left(\frac{1}{w^2}\right)^2 \left(\frac{1}{x^2}\right)^2 \eta_{ij} dx^i dx^j \\ &= \left(\frac{1}{1 - 2x^i b_i + b^2 x^2}\right)^2 \eta_{ij} dx^i dx^j \end{aligned}$$

so the metric transforms according to

$$\eta_{ij} \to \left(1 - 2b^i x_i + b^2 x^2\right)^{-2} \eta_{ij} \tag{10}$$

and the combined transformation is therefore conformal. This time, however, the conformal factor is not the same at every point. These transformations are nonetheless global because the parameters b^i are constant – letting b^i be an arbitrary function of position would enormously enlarge the symmetry in a way that no longer returns a multiple of the metric.

In its usual form, Newton's second law is *not* invariant under global special conformal transformations. The derivatives involved in the acceleration do not commute with the position dependent transformation:

$$e^{-2\lambda(b,x)} \left(\frac{\alpha_0}{v_0}\right) \mathbf{F} \neq e^{-\lambda(x)} \frac{1}{v_0} \frac{d}{dt} \left(e^{-\lambda(b,x)} \frac{\partial x^i}{\partial q^j} \alpha_0 m v^j\right) \tag{11}$$

and the dynamical law is not invariant.

4 A consistent global symmetry for Newtonian mechanics

Before we can gauge any symmetry of Newtonian mechanics, we face the issue described in the second section: our measurement theory and our dynamical equation have different symmetries. The procedure in Newtonian mechanics is to restrict to the intersection of the two symmetries, retaining only global translations and global orthogonal transformations, giving the inhomogeneous orthogonal group, $ISO(3)$. Since $ISO(3)$ lies in the intersection of the symmetries of the dynamical law and the measurement theory, it can be gauged immediately to allow local $SO(3)$ transformations. However, in keeping with our (Goldilocks) principal of maximal symmetry, and noting that the conformal symmetry of the measurement theory is larger than the Euclidean symmetry of the second law, we should gauge $SO(4,1)$. Before we can do this, we must rewrite the second law with global conformal symmetry, $SO(4,1)$. This extension is the subject of the present section. The global conformal symmetry may then be gauged to allow local $SO(3)\times R^{+}$ (homothetic) transformations. In subsequent sections we will carry out both the $ISO(3)$ and $SO(4,1)$ gaugings.

4.1 The conformal connection

Our goal is now to write a form of Newton's second law which is covariant with respect to global conformal transformations. To begin, we have the set of global transformations

$$\begin{aligned}
y^i &= O^i{}_j x^j \\
y^i &= x^i + a^i \\
y^i &= e^{\lambda} x^i \\
y^i &= \frac{x^i + x^2 b^i}{1 + 2b\cdot x + b^2 x^2} = \beta^{-1}\left(x^i + x^2 b^i\right)
\end{aligned}$$

As seen above, it is the derivatives that obstruct the full conformal symmetry (see eq.(11)). The first three transformations already commute with ordinary partial differentiation of tensors because they depend only on the constant parameters $O^i{}_j$, a^i and λ. After a special conformal transformation, however, the velocity becomes a complicated function of position, and when we compute the acceleration,

$$a^i = \frac{dv^i}{dt} = \frac{\partial y^i}{\partial x^j}\frac{d^2 x^i}{dt^2} + v^k \frac{\partial}{\partial x^k}\left(\frac{\partial y^i}{\partial x^j} v^j\right)$$

the result is not only a terrible mess – it is a different terrible mess than what we get from the force (see Appendix 3). The problem is solved if we can find a new derivative operator that commutes with global special conformal transformations.

The mass also poses an interesting problem. If we write the second law as

$$\mathbf{F} = \frac{d}{dt}\left(m\mathbf{v}\right) \tag{12}$$

we see that even "constant" scalars such as mass pick up position dependence and contribute unwanted terms when differentiated

$$\begin{aligned} m &\rightarrow e^{-\lambda(x)}m \\ \partial_i m &\rightarrow e^{-\lambda(x)}\partial_i m - e^{-\lambda(x)} m\partial_i \lambda \end{aligned}$$

We can correct this problem as well, with an appropriate covariant derivative.

To find the appropriate derivation, we consider scalars and vectors, with differentiation of higher rank tensors following by the Leibnitz rule. For scalars of conformal weight n we require

$$D_k s_{(n)} = \partial_k s_{(n)} + n s_{(n)} \Sigma_k$$

while for vectors of weight n we require a covariant derivative of the form,

$$D_k v^i_{(n)} = \partial_k v^i_{(n)} + v^j_{(n)} \Lambda^i_{jk} + n v^i_{(n)} \Sigma_k$$

where Λ^i_{jk} and Σ_k remain to be determined.

Treating the scalar case first, we easily find the required transformation law for Σ_k. Transforming $s_{(n)}$ we demand covariance,

$$D'_k s'_{(n)} = \left(D_k s_{(n)}\right)'$$

where

$$\begin{aligned} s'_{(n)} &= e^{n\lambda} s_{(n)} \\ D'_k s'_{(n)} &= e^{-\lambda}\partial_k \left(e^{n\lambda} s_{(n)}\right) + n\left(e^{n\lambda} s_{(n)}\right)\Sigma'_k \end{aligned}$$

and

$$\left(D_k s_{(n)}\right)' = e^{n'\lambda}\left(D_k s_{(n)}\right)$$

Since derivatives have conformal weight -1, we expect that[1]

$$n' = n - 1$$

Imposing the covariance condition,

$$\begin{aligned} e^{-\lambda}\partial_k \left(e^{n\lambda} s_{(n)}\right) + n\left(e^{n\lambda} s_{(n)}\right)\Sigma'_k &= e^{n'\lambda}\left(D_k s_{(n)}\right) \\ e^{-\lambda}\left(s_{(n)} n \partial_k \lambda + \partial_k s_{(n)}\right) + n s_{(n)} \Sigma'_k &= e^{-\lambda}\left(\partial_k s_{(n)} + n s_{(n)} \Sigma_k\right) \\ e^{-\lambda} s_{(n)} n \partial_k \lambda + n s_{(n)} \Sigma'_k &= e^{-\lambda} n s_{(n)} \Sigma_k \end{aligned}$$

or since this must hold for all $s_{(n)}$,

$$\Sigma'_k = e^{-\lambda}\left(\Sigma_k - \partial_k \lambda\right)$$

[1] In field theory, the coordinates and therefore the covariant derivative are usually taken to have zero weight, while dynamical fields and the metric carry the dimensional information. In Newtonian physics, however, the coordinate of a particle is a dynamical variable, and must be assigned a weight.

Since we assume the usual form of Newton's law holds in some set of coordinates, Σ_k will be zero for these coordinate systems. Therefore, we can take Σ_k to be zero until we perform a special conformal transformation, when it becomes $-ne^{-\lambda}\partial_k\lambda$. Notice that since λ is constant for a dilatation, Σ_k remains zero if we simply change from furlongs to feet.

Since a special conformal transformation changes the metric from the flat metric η_{ij} to the conformal metric

$$g_{ij} = e^{2\lambda(x)}\eta_{ij} = \beta^{-2}\eta_{ij} \tag{13}$$

where

$$\beta = 1 + 2\mathbf{b}\cdot\mathbf{x} + b^2x^2 \tag{14}$$

we need a connection consistent with a very limited set of coordinate transformations. This just leads to a highly restricted form of the usual metric compatible Christoffel connection. From eq.(13) we compute immediately,

$$\begin{aligned}\Lambda^i_{jk} &= \frac{1}{2}g^{im}\left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right) \\ &= \eta^{im}\left(\eta_{mj}\lambda_{,k} + \eta_{mk}\lambda_{,j} - \eta_{jk}\lambda_{,m}\right)\end{aligned} \tag{15}$$

where

$$\lambda_{,k} = -\beta^{-1}\beta_{,k}$$

Notice that Λ^i_{jk} has conformal weight -1, and vanishes whenever $b_i = 0$.

We can relate Σ_k directly to the special conformal connection Λ^i_{jk}. The trace of Λ^i_{jk} is

$$\Lambda_k \equiv \Lambda^i_{ik} = 3\lambda_{,k}$$

so that

$$\begin{aligned}\Sigma_k &= -\lambda_{,k} \\ &= -\frac{1}{3}\Lambda_k\end{aligned}$$

The full covariant derivative of a vector of conformal weight n, may therefore be written as

$$D_k v^i_{(n)} = \partial_k v^i_{(n)} + v^j_{(n)}\left(\Lambda^i_{jk} - \frac{n}{3}\Lambda_k\delta^i_j\right) \tag{16}$$

where Λ^i_{jk} is given by eq.(15).

4.2 Covariantly constant masses

Extending the symmetry of classical mechanics to include special conformal transformations introduces an unusual feature: even constants such as mass may appear to be position dependent. But we are now in a position to say what it means for a scalar to be constant. Since mass has conformal weight -1, we demand

$$D_k m = \partial_k m + \frac{1}{3}\Lambda_k m = 0$$

That is, constant mass now means covariantly constant mass.

This equation is always integrable because Λ_k is curl-free,

$$\Lambda_{k,m} - \Lambda_{m,k} = 3\left(\lambda_{,km} - \lambda_{,mk}\right) = 0$$

Integrating,

$$m = m_0 e^{-\lambda}$$

Any set of N masses, $\left\{m_{(1)}, m_{(2)}, \ldots, m_{(N)}\right\}$, in which each element satisfies the same condition,

$$D_k m_{(i)} = 0, \qquad i = 1, \ldots, N$$

gives rise to an *invariant spectrum* of $N-1$ measurable mass ratios,

$$M_R = \left\{\frac{m_1}{m_1} = 1, \frac{m_2}{m_1}, \ldots, \frac{m_N}{m_1}\right\}$$

since the conformal factor cancels out. Here we have arbitrarily chosen $\frac{1}{\alpha_0 v_0 m_1}$ as our unit of length.

4.3 The conformally covariant second law

We can also write Newton's second law in a covariant way. The force is a weight -2 vector. With the velocity transforming as a weight zero vector and the mass as a weight -1 scalar, the time derivative of the momentum now requires a covariant derivative,

$$\frac{D\left(mv^i\right)}{Dt} = \frac{d}{dt}\left(mv^i\right) + mv^j v^k \Lambda^i_{jk} + \frac{1}{3} mv^i v^k \Lambda_k$$

Then Newton's law is

$$F^i = \frac{D}{Dt}\left(mv^i\right)$$

To see how this extended dynamical law transforms, we check conformal weights. The velocity has the dimensionless form

$$\frac{1}{v_0}\frac{dx^i}{dt}$$

The covariant derivative reduces this by one, so the acceleration has conformal weight -1. The mass also has weight -1, while the force, as noted above, has weight -2. Then we have:

$$\tilde{F}^i = \frac{D}{D\tilde{t}}\left(\tilde{m}\tilde{v}^i\right)$$

The first term in the covariant time derivative becomes

$$\begin{aligned}
\frac{d}{d\tilde{t}}\left(\tilde{m}\tilde{v}^i\right) &= e^{-\lambda}\frac{d}{dt}\left(e^{-\lambda} m \frac{\partial y^i}{\partial x^j} v^j\right) \\
&= e^{-2\lambda}\frac{\partial y^i}{\partial x^j}\frac{d}{dt}\left(mv^j\right) + e^{-\lambda} mv^j \frac{dx^k}{dt}\frac{\partial}{\partial x^k}\left(e^{-\lambda}\frac{\partial y^i}{\partial x^j}\right)
\end{aligned}$$

The final term on the right exactly cancels the inhomogeneous contributions from Λ^i_{jk} and Λ_k, leaving the same conformal factor and Jacobian that multiply the force:

$$e^{-2\lambda}\frac{\partial y^i}{\partial x^j}F^j = e^{-2\lambda}\frac{\partial y^i}{\partial x^j}\left(\frac{d\left(mv^j\right)}{dt} + mv^m v^k \Lambda^j_{mk} + \frac{1}{3}mv^j v^k \Lambda_k\right)$$

The conformal factor and Jacobian cancel, so if the globally conformally covariant Newton's equation holds in one conformal frame, it holds in all conformal frames.

The transformation to the conformally flat metric

$$g_{ij} = e^{2\lambda}\eta_{ij} = \beta^{-2}\eta_{ij}$$

does not leave the curvature tensor invariant. This only makes sense – just as we have an equivalence class of metrics, we require an equivalence class of curved spacetimes. Though the curvature for g_{ij} is well known (see, eg. Hawking and Ellis [6]) we provide the simple calculation in Appendix 4. Since the acceleration is the variation of the line element, force-free motion is represented by geodesics in any of these physically equivalent geometries.

We now want to consider what happens when we gauge the symmetries associated with classical mechanics. In the next section, we outline some basics of gauge theory. Then in succeeding sections we consider two gauge theories associated with Newtonian mechanics. First, we gauge the Euclidean $ISO(3)$ invariance of $F^i = ma^i$, then the full $SO(4,1)$ conformal symmetry of $F^i = \frac{D}{Dt}\left(mv^i\right)$.

Before performing these gaugings, we digress to describe the quotient group method of gauging.

5 Gauge theory

Here we briefly outline the quotient group method of gauging a symmetry group. For internal symmetries such as the $U(1)$ symmetry of electromagnetism the quotient method may be used, but there are simpler techniques. However, for gravitational or other gauge theories that involve construction of a physical space the quotient method is necessary. The method may be used, for example, to construct the Riemannian geometries of general relativity from the quotient of the Poincaré group by its Lorentz subgroup. We require a similar construction of Euclidean 3-space and a symplectic 6-space for $ISO(3)$ and $SO(4,1)$, respectively.

The general case begins with a Lie group, $\mathcal{G}$, and its Lie algebra

$$[G_A, G_B] = c_{AB}{}^{C}G_C$$

Suppose further that $\mathcal{G}$ has a subgroup $\mathcal{H}$, such that $\mathcal{H}$ itself has no subgroup normal in $\mathcal{G}$. Then the quotient group $\mathcal{G}/\mathcal{H}$ is a manifold with the symmetry $\mathcal{H}$ acting independently at each point (technically, a fiber bundle). $\mathcal{H}$ is now called the isotropy subgroup. The manifold inherits a connection from the original group, so we know how to take $\mathcal{H}$-covariant derivatives. We may then generalize both the manifold and the connection, to arrive at a class of manifolds with curvature, still having local $\mathcal{H}$ symmetry. We consider here only the practical application of the method. Full mathematical details may be found, for example, in [7], [8], [9].

The connection is developed as follows. Rewriting the Lie algebra in the dual basis of Lie algebra valued 1-forms defined by

$$\left\langle G_A, \omega^B \right\rangle = \delta_A^B$$

we find the Maurer-Cartan equation for $\mathcal{G}$,

$$\mathbf{d}\omega^C = -\frac{1}{2} c_{AB}{}^C \omega^A \wedge \omega^B$$

This is fully equivalent to the Lie algebra above, with $\mathbf{d}^2 = 0$ giving the Jacobi identity. Now consider the quotient of $\mathcal{G}$ by $\mathcal{H}$. The Maurer-Cartan equation has the same appearance, except that now all of the connection 1-forms ω^A are regarded as linear combinations of a smaller set spanning the cotangent space of the quotient manifold.

In slightly more detail, let the Lie algebra of $\mathcal{H}$ have commutators

$$[H_a, H_b] = c_{ab}{}^c H_c$$

then the Lie algebra for $\mathcal{G}$ may be written as

$$\begin{aligned}
[G_\alpha, G_\beta] &= c_{\alpha\beta}{}^\rho G_\rho + c_{\alpha\beta}{}^a H_a \\
[G_\alpha, H_a] &= c_{\alpha a}{}^\rho G_\rho + c_{\alpha a}{}^b H_b \\
[H_a, H_b] &= c_{ab}{}^c H_c
\end{aligned}$$

where α and a together span the full range of the indices A. Because $\mathcal{H}$ contains no normal subgroup of $\mathcal{G}$, the constants $c_{\alpha a}{}^\rho$ are nonvanishing for some α for all a. The Maurer-Cartan structure equations take the corresponding form

$$\mathbf{d}\omega^\rho = -\frac{1}{2} c_{\alpha\beta}{}^\rho \omega^\alpha \wedge \omega^\beta - \frac{1}{2} c_{\alpha a}{}^\rho \omega^\alpha \wedge \omega^a \tag{17}$$

$$\mathbf{d}\omega^a = -\frac{1}{2} c_{\alpha\beta}{}^{a\alpha} \wedge \omega^\beta - \frac{1}{2} c_{\alpha b}{}^a \omega^\alpha \wedge \omega^b - \frac{1}{2} c_{bc}{}^a \omega^b \wedge \omega^c \tag{18}$$

and we regard the forms ω^a as linearly dependent on the ω^α,

$$\omega^a = \omega^a{}_\alpha \omega^\alpha$$

The forms ω^α span the spaces cotangent to the base manifold and the ω^a give an $\mathcal{H}$-symmetric connection.

Of particular interest for our formulation is the fact that eq.(17) gives rise to a covariant derivative. Because $\mathcal{H}$ is a subgroup, $\mathbf{d}\omega^\rho$ contains no term quadratic in ω^a, and may therefore be used to write

$$0 = \mathbf{D}\omega^\rho \equiv \mathbf{d}\omega^\rho + \omega^\alpha \wedge \omega_\alpha{}^\rho$$

with

$$\omega_\alpha{}^\rho \equiv \frac{1}{2}c_{\alpha\beta}{}^\rho\omega^\beta - \frac{1}{2}c_{\alpha a}{}^\rho\omega^a$$

This expresses the covariant constancy of the basis. As we shall see in our $SO(3)$ gauging, this derivative of the orthonormal frames ω^ρ is not only covariant with respect to local $\mathcal{H} = SO(3)$ transformations, but also leads directly to a covariant derivative with respect to general coordinate transformations when expressed in a coordinate basis. This is the reason that general relativity may be expressed as both a local Lorentz gauge theory and a generally coordinate invariant theory, and it is the reason that Lagrangian mechanics with its "generalized coordinates" may also be written as a local $SO(3)$ gauge theory.

Continuing with the general method, we introduce curvature by changing the connection. This means that the Maurer-Cartan equations are no longer satisfied, but gain additional terms,

$$\begin{aligned}
\mathbf{d}\omega^\rho &= -\frac{1}{2}c_{\alpha\beta}{}^\rho\omega^\alpha\wedge\omega^\beta - \frac{1}{2}c_{\alpha a}{}^\rho\omega^\alpha\wedge\omega^a + \mathbf{R}^\rho \\
\mathbf{d}\omega^a &= -\frac{1}{2}c_{\alpha\beta}{}^{a\alpha}\wedge\omega^\beta - \frac{1}{2}c_{\alpha b}{}^a\omega^\alpha\wedge\omega^b - \frac{1}{2}c_{bc}{}^a\omega^b\wedge\omega^c + \mathbf{R}^a
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\mathbf{R}^a &= \frac{1}{2}R^a{}_{\alpha\beta}\omega^\alpha\wedge\omega^\beta \\
\mathbf{R}^\rho &= \frac{1}{2}R^\rho{}_{\alpha\beta}\omega^\alpha\wedge\omega^\beta
\end{aligned}$$

are 2-forms. These 2-forms are quadratic in the basis forms ω^α if and only if they describe curvature of the quotient manifold and not the entire original group. In a physical theory, $\mathbf{R}^a$ and $\mathbf{R}^\rho$ are specified by some set of field equations, and the modified connection is found by solving eqs.(19). Since eqs.(19) describe only local structure, we may allow any manifold consistent with the modified connection.

We illustrate with the Poincaré group. The quotient of the Poincaré group by the Lorentz group is the manifold R^4. The generators of the Poincaré Lie algebra satisfy

$$\begin{aligned}
[M^a{}_b, M^c{}_d] &= -\frac{1}{2}\left(\delta^c_b M^a{}_d - \eta^{ac}M_{bd} - \eta_{bd}M^{ac} + \delta^a_d M_b{}^c\right) \\
[M^a{}_b, P^c] &= \frac{1}{2}\left(\eta^{ac}P_b - \delta^c_b P^a\right) \\
\left[P^a, P^b\right] &= 0
\end{aligned}$$

where the Lorentz subgroup is generated by the $M^a{}_b$. Defining dual 1-forms $\omega^a{}_b$ and $\mathbf{e}^a$, the Maurer-Cartan structure equations take the form

$$\begin{aligned}
\mathbf{d}\omega^a{}_b &= \omega^c{}_b\wedge\omega^a{}_c \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\wedge\omega^a{}_b
\end{aligned}$$

and regarding the connection forms $\omega^a{}_b$ as linear combinations of the cotangent basis $\mathbf{e}^a$,

$$\omega^a{}_b = \omega^a{}_{bc}\mathbf{e}^c$$

the system describes a local Lorentz connection on Minkowski spacetime. The connection forms $\omega^a{}_b$ comprise the spin connection and the set of basis forms $\mathbf{e}^a$ is called the solder form. By changing the connection (and the manifold, if desired), the Maurer-Cartan equations generalize to include the Riemann curvature 2 -form, $\mathbf{R}^a{}_b$, and the torsion 2-form, $\mathbf{T}^a$,

$$\begin{aligned}
\mathbf{d}\omega^a{}_b &= \omega^c{}_b \wedge \omega^a{}_c + \mathbf{R}^a{}_b \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \omega^a{}_b + \mathbf{T}^a
\end{aligned}$$

If the torsion is zero, these equations describe an arbitrary Riemannian geometry. General relativity follows by setting $\mathbf{T}^a = 0$ and imposing the Einstein equation on $\mathbf{R}^a{}_b$. The metric may be found algebraically from the components of $\mathbf{e}^a$.

Our gaugings of Newtonian theory below will further illustrate the method, although we will not generalize to curved spaces or different manifolds. As a result, the structure equations in the form of eqs.(17, 18) describe the geometry and symmetry of our gauged dynamical law.

6 A Euclidean gauge theory of Newtonian mechanics

We begin by gauging the usual restricted form of the second law, using the Euclidean group as the initial global symmetry. Just as gauging the Poincaré group of flat spacetime leads to the generally coordinate invariant arena for general relativity, the result of the Euclidean gauging is the general coordinate invariant form of Newton's law, i.e., Lagrangian mechanics. While the result is not in itself surprising, it provides a new route to familiar results. More importantly, it shows our method of construction in a familiar context, before we apply it to conformal transformations and find an unexpected result.

The familiar form (see Appendix 5) of the Lie algebra of the Euclidean group, $iso(3)$, is

$$\begin{aligned}
[J_i, J_j] &= \varepsilon_{ij}{}^k J_k \\
[J_i, P_j] &= \varepsilon_{ij}{}^k P_k \\
[P_i, P_j] &= 0
\end{aligned}$$

Using the quotient group method, we choose $so(3)$ as the isotropy subgroup. Then introducing the Lie algebra valued 1-forms ω^i dual to J_i and $\mathbf{e}^i$ dual to P_i we write the Maurer-Cartan structure equations

$$\begin{aligned}
\mathbf{d}\omega^m &= -\frac{1}{2}c_{ij}{}^m\omega^i\omega^j = -\frac{1}{2}\varepsilon_{ij}{}^m\omega^i\omega^j \\
\mathbf{d}\mathbf{e}^m &= -c_{ij}{}^m\omega^i\mathbf{e}^j = -\varepsilon_{ij}{}^m\omega^i\mathbf{e}^j
\end{aligned}$$

Defining

$$\begin{aligned}\omega^{mn} &\equiv \omega^k \varepsilon_k^{\ \ mn} \\ \omega^k &= \frac{1}{2}\varepsilon^k_{\ \ mn}\omega^{mn}\end{aligned}$$

these take a form similiar to the structure equations for general relativity,

$$\mathbf{d}\omega^{mn} = \omega^{mk}\omega_k^{\ \ n} \tag{20}$$

$$\mathbf{d}\mathbf{e}^m = \mathbf{e}^k\omega_k^{\ \ m} \tag{21}$$

with ω^{mn} the spin connection and $\mathbf{e}^m$ the dreibein. These equations are equivalent to the commutation relations of the Lie algebra, with the Jacobi identity following as the integrability condition $\mathbf{d}^2 = 0$, i.e.,

$$\begin{aligned}\mathbf{d}^2\omega^{mn} &= \mathbf{d}\left(\omega^{mj}\omega_j^{\ \ n}\right) = \mathbf{d}\omega^{mj}\omega_j^{\ \ n} - \omega^{mj}\mathbf{d}\omega_j^{\ \ n} \equiv 0 \\ \mathbf{d}^2\mathbf{e}^m &= \mathbf{d}\left(\omega^m_{\ \ k}\mathbf{e}^k\right) = \mathbf{d}\omega^m_{\ \ k}\mathbf{e}^k - \omega^m_{\ \ k}\mathbf{d}\mathbf{e}^k \equiv 0\end{aligned}$$

Eqs.(20) and (21) define a connection on a three dimensional (flat) manifold spanned by the three 1-forms $\mathbf{e}^m$. We take ω^{mn} to be a linear combination of the $\mathbf{e}^m$. This completes the basic construction.

The equations admit an immediate solution because the spin connection, ω^{mn} is in involution. The 6-dimensional group manifold therefore admits coordinates y^i such that

$$\omega^{mn} = w^{mn}_{\quad\ \alpha}\mathbf{d}y^\alpha$$

Here we use Latin indices for indices in the orthonormal basis $\mathbf{e}^m$ and Greek indices for the coordinate basis. By the Frobenius theorem, there are submanifolds given by $y^\alpha = const.$ On these 3-dimensional submanifolds, $\omega^{mn} = 0$ and therefore

$$\mathbf{d}\mathbf{e}^m = \omega^m_{\ \ k}\mathbf{e}^k = 0$$

with solution

$$\mathbf{e}^m = \delta^m_\alpha \mathbf{d}x^\alpha$$

for an additional three coordinate functions x^α. This solution gives Cartesian coordinates on the $y^\alpha = const.$ submanifolds. Identifying these manifolds as copies of our Euclidean 3-space, we are now free to perform an arbitrary rotation at each point.

Performing such local rotations on orthonormal frames leads us to general coordinate systems. When we do this, the spin connection ω^{mn} takes the pure gauge form

$$\omega^{mn} = -\left(\mathbf{d}O^m_{\ \ j}\right)\bar{O}^{jn}$$

where $O^m_{\ \ j}(x)$ is a local orthogonal transformation and $\bar{O}^{jn}(x)$ its inverse. Then $\mathbf{e}^i$ provides a general orthonormal frame field in the locally rotated basis,

$$\mathbf{e}^i = e_\alpha^{\ \ i}\mathbf{d}x^\alpha$$

The coefficients $e_\alpha^{\ \ i}(x)$ may be determined once we know $O^m_{\ \ j}(x)$.

The second Maurer-Cartan equation gives us a covariant derivative as follows. Expand any 1-form in the orthonormal basis,

$$\mathbf{v} = v_i \mathbf{e}^i$$

Then we define the covariant exterior derivative via

$$\begin{aligned}
(\mathbf{D}v_i)\,\mathbf{e}^i &= \mathbf{dv} \\
&= \mathbf{d}\left(v_i \mathbf{e}^i\right) \\
&= \mathbf{d}v_i \mathbf{e}^i + v_i \mathbf{de}^i \\
&= \left(\mathbf{d}v_k - v_i \omega_k^{\ \ i}\right)\ \mathbf{e}^k
\end{aligned}$$

Similar use of the product rule gives the covariant derivative of higher rank tensors. This local $SO(3)$-covariant derivative of forms in an orthonormal basis is equivalent to a general coordinate covariant derivative when expressed in terms of a coordinate basis. We see this as follows.

Rewriting eq.(21) in the form

$$\mathbf{de}^i + \mathbf{e}^k \omega^i_{\ \ k} = 0$$

we expand in an arbitrary coordinate basis, to find

$$\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \left(\partial_\alpha e_\beta^{\ \ i} + e_\alpha^{\ \ k} \omega^i_{\ \ k\beta}\right) = 0$$

The term in parentheses must therefore be symmetric:

$$\partial_\alpha e_\beta^{\ \ i} + e_\alpha^{\ \ k} \omega^i_{k\beta} \equiv \Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$$

Writing

$$\Gamma^i_{\beta\alpha} = e_\mu^{\ \ i} \Gamma^\mu_{\beta\alpha}$$

we define the covariant constancy of the basis coefficients,

$$D_\alpha e_\beta^{\ \ i} \equiv \partial_\alpha e_\beta^{\ \ i} + e_\alpha^{\ \ k} \omega^i_{k\beta} - e_\mu^{\ \ i} \Gamma^\mu_{\beta\alpha} = 0 \qquad (22)$$

Eq.(22) relates the $SO(3)$-covariant spin connection for orthonormal frames to the Christoffel connection for general coordinate transformations. Next, note that the covariant derivative of the orthogonal metric $\eta = diag(1,1,1)$ is zero,

$$\begin{aligned}
D_\alpha \eta_{ab} &= \partial_\alpha \eta_{ab} - \eta_{cb a}^{\ \ c} - \eta_{ac} \omega_b^c \\
&= -\eta_{cb} \omega_a^c - \eta_{ac} \omega_b^c \\
&= 0
\end{aligned}$$

where the last step follows by the antisymmetry of the $SO(3)$ connection. Since the inverse orthogonal metric η^{ab} is given by the linear inner product of two basis 1-forms, we have

$$\begin{aligned}
\eta_{ab} &= \mathbf{e}^a \cdot \mathbf{e}^b \\
&= e_\alpha^{\ \ a} e_\beta^{\ \ b} \mathbf{d}x^\alpha \cdot \mathbf{d}x^\beta
\end{aligned}$$

Let the inverse to $e_\alpha^{\ \ a}$ be written as $e_a^{\ \ \alpha}$, and write the inverse coordinate metric as the inner product of coordinate basis forms

$$g^{\alpha\beta} = \mathbf{d}x^\alpha \cdot \mathbf{d}x^\beta$$

Then we have the relationship between the coordinate and orthonormal forms of the inverse metric and metric,

$$\begin{aligned} g^{\alpha\beta} &= e_a^{\ \ \alpha} e_b^{\ \ \beta} \eta^{ab} \\ g_{\alpha\beta} &= \eta_{ab} e_\alpha^{\ \ a} e_\beta^{\ \ b} \end{aligned}$$

The covariant constancy of the coordinate metric follows immediately,

$$D_\alpha g_{\mu\nu} = D_\alpha \left(\eta_{ab} e_\mu^{\ \ a} e_\nu^{\ \ b}\right) = 0$$

This is inverted in the usual way to give the Christoffel connection for $SO(3)$,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right) \tag{23}$$

Thus, our solution for the solder form and spin connection $(\mathbf{e}^a, \omega^a_{\ \ b})$ lead us to the Chrostoffel connection, explicitly establishing the relationship between diffeomorphism invariance and local $SO\,(3)$ invariance. The Christoffel connection may also be found directly from eq.(22) using $g_{\mu\nu} = \eta_{ab} e_\mu^{\ \ a} e_\nu^{\ \ b}$. Thus, there is little practical difference between the ability to perform local rotations on an orthonormal frame field, and the ability to perform arbitrary transformations of coordinates. It is just a matter of putting the emphasis on the coordinates or on the basis vectors (see [10]). It is this equivalence that makes the $SO(3)$ gauge theory equivalent to the use of "generalized coordinates" in Lagrangian mechanics.

Since Newtonian 3-space is Euclidean and we have not generalized to curved spaces, the metric is always just a diffeomorphism away from orthonormal, that is,

$$\begin{aligned} e_\alpha^{\ \ a} &= J_\alpha^{\ \ a} = \frac{\partial y^a}{\partial x^\alpha} \\ g_{\alpha\beta} &= \eta_{ab} \mathbf{e}_\alpha^{\ \ a} \mathbf{e}_\beta^{\ \ b} = \eta_{ab} \frac{\partial y^a}{\partial x^\alpha} \frac{\partial y^b}{\partial x^\beta} \end{aligned} \tag{24}$$

and the connection takes the simple form

$$\Gamma^\alpha_{\mu\nu} = -\frac{\partial x^\alpha}{\partial y^a} \frac{\partial^2 y^a}{\partial x^\mu \partial x^\nu} \tag{25}$$

which has, of course, vanishing curvature. Notice that $(\mathbf{e}^a, \omega^a_{\ \ c})$ or equivalently, $\left(g_{\alpha\beta}, \Gamma^\alpha_{\mu\nu}\right)$, here describe a much larger class of coordinate transformations than the global conformal connection Λ^i_{jk} of Sec. 4. The connection of eq.(25) gives a derivative which is covariant for *any* coordinate transformation.

This completes our description of Euclidean 3-space in local $SO(3)$ frames or general coordinates. We now generalize Newton's second law to be consistent with this enhanced symmetry.

6.1 Generally covariant form of Newton's law

The generalization of Newton's second law to a locally $SO(3)$ covariant form of mechanics is now immediate. We need only replace the time derivative by a directional covariant derivative,

$$F^{\alpha} = v^{\beta} D_{\beta} \left(m v^{\alpha} \right)$$

where

$$D_{\beta} v^{\alpha} \equiv \partial_{\beta} v^{\alpha} + v^{\mu} \Gamma^{\alpha}_{\mu\beta} \tag{26}$$

and $\Gamma^{\alpha}_{\mu\beta}$ given by eq.(25). This is the principal result of the the $SO(3)$ gauging.

If F^{α} is curl free, then it may be written as minus the contravariant form of the gradient of a position-dependent potential, $V\left(x^{\alpha}\right)$,

$$F^{\alpha} = -g^{a\beta} \frac{\partial V}{\partial x^{\beta}}$$

and the covariant second law may be written as

$$v^{\beta} \partial_{\beta} \left(m v^{\alpha} \right) + m v^{\mu} v^{\beta} \Gamma^{\alpha}_{\mu\beta} = -g^{a\beta} \frac{\partial V}{\partial x^{\beta}}$$

This result agrees with that of [5].

Continuing, we expand the connection in terms of the metric,

$$\begin{aligned}
v^{\beta} \partial_{\beta} \left(m v^{\alpha} \right) + m v^{\mu} v^{\beta} \Gamma^{\alpha}_{\mu\beta} &= -g^{a\beta} \frac{\partial V}{\partial x^{\beta}} \\
v^{\beta} \partial_{\beta} \left(m v^{\alpha} \right) + \frac{1}{2} m v^{\mu} v^{\beta} g^{\alpha\nu} \left(g_{\nu\mu,\beta} + g_{\nu\beta,\mu} - g_{\mu\beta,\nu} \right) &= -g^{a\beta} \frac{\partial V}{\partial x^{\beta}} \\
g^{\alpha\nu} g_{\nu\mu} v^{\beta} \partial_{\beta} \left(m v^{\mu} \right) + m v^{\mu} v^{\beta} g^{\alpha\nu} g_{\nu\mu,\beta} - \frac{1}{2} m v^{\mu} v^{\beta} g^{\alpha\nu} g_{\mu\beta,\nu} &= -g^{a\beta} \frac{\partial V}{\partial x^{\beta}} \\
v^{\beta} \partial_{\beta} \left(m g_{\nu\mu} v^{\mu} \right) - \frac{1}{2} v^{\mu} v^{\beta} g_{\mu\beta,\nu} &= -\frac{\partial V}{\partial x^{\nu}} \\
v^{\beta} \partial_{\beta} \left(m g_{\nu\mu} v^{\mu} \right) &= \frac{\partial}{\partial x^{\nu}} \left(\frac{1}{2} v^{\mu} v^{\beta} g_{\mu\beta} - V \right)
\end{aligned}$$

Defining the kinetic energy

$$T = \frac{1}{2} m g_{\alpha\beta} v^{\alpha} v^{\beta}$$

and recognizing that

$$\frac{\partial T}{\partial v^{\nu}} = m g_{\nu\mu} v^{\mu}$$

the diffeomorphism invariant form of the second law may be written as

$$\frac{d}{dt} \left(\frac{\partial T}{\partial v^{\nu}} \right) = \frac{\partial}{\partial x^{\nu}} \left(T - V \right)$$

Finally, since the potential is independent of the velocity, we may set

$$L = T - V$$

to get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0$$

This, of course, is the Euler-Lagrange equation. This argument provides a derivation of the usual form of the classical Lagrangian,

$$L = T - V = \frac{1}{2} m g_{jk} v^j v^k - V$$

from the gauge principle, and shows that the covariant form of the law is the extremum of a functional,

$$S = \int (T - V)\, dt$$

Thus, $ISO(3)$ gauge theory has led us to Lagrangian mechanics. The derivation of the form of the classical Lagrangian and its variational character as consequences of gauge theory are the central results of this section. These results are expected since Lagrangian mechanics was formulated in order to allow "generalized coordinates", i.e., diffeomorphism covariant equations of motion.

6.2 Multiple particles

We now generalize these results to multiple particles. In the standard treatment, the Lagrangian for many particles is the sum of the individual single-particle kinetic terms together with the multiparticle potential. We show that the same result follows from gauge theory. We conclude with an amusing proof regarding the additivity of the multiparticle action.

6.2.1 Generalization to multiple particles

To treat the case of multiple particles, we may start again with Newton's second law, but this time we assume there are N particles. The forces on the various particles arise as the gradient of a potential V which may depend on the positions of all N particles, $V = V\left(x^\alpha_{(1)}, \ldots, x^\beta_{(N)}\right)$. Therefore, with $A = 1, \ldots, N$, the forces may be written covariantly as

$$F^a_{(A)} = -g^{ab} \frac{\partial V}{\partial x^b_{(A)}}$$

As we showed above, the acceleration of the A^{th} particle is written covariantly as

$$a^\alpha_{(A)} = v^\beta_{(A)} \partial_\beta v^\alpha_{(A)} + v^\mu_{(A)} v^\beta_{(A)} \Gamma^\alpha_{\mu\beta}$$

where $\Gamma^{\alpha}_{\mu\beta}$ is evaluated at the position of the A^{th} particle. Therefore,

$$v^{\beta}_{(A)}\partial_{\beta}m_{(A)}v^{\alpha}_{(A)} + m_{(A)}v^{\mu}_{(A)}v^{\beta}_{(A)}\Gamma^{\alpha}_{\mu\beta}\left(x^{\nu}_{(A)}\right) = -g^{ab}\frac{\partial V}{\partial x^{b}_{(A)}}$$

The argument proceeds exactly as before with the result that for each particle, the diffeomorphism invariant form of the second law may be written as

$$\frac{d}{dt}\left(\frac{\partial\left(T_{(A)} - V\right)}{\partial v^{\nu}_{(A)}}\right) = \frac{\partial}{\partial x^{\nu}_{A}}\left(T_{(A)} - V\right)$$

where

$$T_{(A)} = \frac{1}{2}m_{(A)}g_{\alpha\beta}\left(x^{\nu}_{(A)}\right)v^{\alpha}_{(A)}v^{\beta}_{(A)}$$

These equations of motion are the variational equations for the action functional

$$S = \int\left(\sum_{A=1}^{N}T_{(A)} - V\left(x_1, \ldots, x_N\right)\right)dt$$

where each coordinate vector $x^{\alpha}_{(A)}$ is varied independently.

6.2.2 Additivity of the multiparticle action

We conclude our discussion of $SO\left(3\right)$ gauging with a theorem. The multiparticle action includes a sum over the separate kiinetic energies of the particles, but there is only a single potential. This means that the actions for distinct particles are not additive. Is it possible to reformulate our variational principle as a sum over single particle actions?

We answer this question in the affirmative. First, we see that the requisite potentials exist as follows. Suppose we want an appropriate potential for particle 1. We can, in principle, solve the equations of motion for the remaining $N-1$ particles, giving functions $x^{\alpha}_{(A)}\left(t\right)$ for $A = 2, \ldots, N$. Substituting these functions into the action we have

$$S = \int dt\left(\frac{m_1}{2}g_{mn}\left(x_1\right)\frac{dx^m_1}{dt}\frac{dx^n_1}{dt} + \sum_{A=2}^{N}\frac{m_A}{2}g_{mn}\left(x_A\left(t\right)\right)\frac{dx^m_A\left(t\right)}{dt}\frac{dx^n_A\left(t\right)}{dt} - V\left(x_1, t\right)\right)$$

Since the middle term is now a function of t alone, it does not contribute to the equation of motion for x^{α}_1 so S is equivalent to

$$S_1 = \int dt\left(T_1 - V_1\left(x_1, t\right)\right)$$

where $V_1 = V\left(x_1, t\right) = V_1\left(x_1, x_2\left(t\right), \ldots, x_N\left(t\right), t\right)$. The potential now only depends on x_1 and time. In the same way we can find separate time-dependent potentials, $V_A\left(x_A, t\right)$, for each particle.

The N-particle action may now be written as the sum

$$S' = \sum_A \int dt \left(T_A - V_A\left(x_A, t\right)\right)$$

Conversely, suppose we are given a set of separate Lagrangians,

$$L_A = T_A - V_A\left(x_A, t\right)$$

Then, with the usual Newtonian assumption of impenetrability, we observe that the world lines of the N particles are non-intersecting. Therefore, at any time t there exist disjoint open neighborhoods, N_A, such that N_A contains the A^{th} particle and such that the closure of the sets N_A remain disjoint,

$$\bar{N}_A \cap \bar{N}_B = \phi$$

Now extend each N_A to an open set U_A such that

1. The sets U_A form an open cover
2. Each set N_A intersects exactly one U_A,

$$U_A \cap N_B = \delta_{AB} N_B$$

Finally, define a partition of unity on the open cover U_A, choosing each f_A such that

$$f_A\left(N_A\right) = 1$$

This condition is clearly compatible with the requirement that f_A be of compact support on U_A. We may now define

$$V\left(x_1^\alpha, \ldots, x_N^\beta, t\right) = \sum_A f_A V_A\left(x_A^\alpha, t\right)$$

This gives the required single potential. We conclude that, for ideal Newtonian particles, the action may be written as a sum of single particle actions if and only if it can be written using a single potential dependent on all of the coordinates and time.

We may strengthen this result by considering a second question. Noting that a single, time-independent potential $V\left(x_1^\alpha, \ldots, x_2^\beta\right)$ will generally give rise to a set of time-dependent individual potentials, $V_A\left(x_A^a, t\right)$, we ask the converse: When does a given set of time-dependent potentials $V_A\left(x_A^\alpha, t\right)$ give rise to a time-independent single potential? For motions with bounded velocity (i.e., essentially all classical physical motions) the answer is surprisingly simple. Let the x-component of the velocity of particle 1 be bounded below by v_0. Then a Galilean boost in the x-direction by $-2v_0$ insures that the x-component of x_1^α is a monotonic function of t. Inverting this function, we may replace the time dependence by additional dependence on x_1^1, achieving the desired result, $V\left(x_1^\alpha, \ldots, x_N^\beta, t\left(x_A^a\right)\right)$. We summarize these results with:

Theorem For ideal Newtonian particles, the action may be written as a sum of single particle actions if and only if it can be written using a single, time-independent potential, $V\left(x_1^\alpha, \ldots, x_N^\beta\right)$.

In the next Section we our construction of the Hamiltonian formulation automatically makes it additive.

7 A conformal gauge theory of Newtonian mechanics

Now we gauge the full $O(4,1)$ symmetry of our globally conformal form of Newton's law. The Lie algebra of the conformal group (see Appendix 5) is:

$$
\begin{aligned}
[M^a{}_b, M^c{}_d] &= \delta_b^c M^a{}_d - \eta^{ca}\eta_{be} M^e{}_d - \eta_{bd} M^{ac} + \delta_d^a M_b{}^c \\
[M^a{}_b, P_c] &= \eta_{bc}\eta^{ae} P_e - \delta_c^a P_b \\
[M^a{}_b, K^c] &= \delta_b^c K^a - \eta^{ca}\eta_{be} K^e \\
[P_b, K_d] &= -\eta_{be} M^e{}_d - \eta_{bd} D \\
[D, P_a] &= -P_a \\
[D, K^a] &= K^a
\end{aligned} \qquad (27)
$$

where $M^a{}_b, P_a, K_a$ and D generate rotations, translations, special conformal transformations and dilatations, respectively.

As before, we write the Lie algebra in terms of the dual basis of 1-forms, setting

$$
\begin{aligned}
\langle M^a{}_b, \omega^c{}_d\rangle &= \delta_b^c \delta_d^a - \eta^{ca}\eta_{be} \\
\langle P_b, \mathbf{e}^a\rangle &= \delta_b^a \\
\langle K^a, \mathbf{f}_b\rangle &= \delta_b^a \\
\langle D, \mathbf{W}\rangle &= 1
\end{aligned}
$$

The Maurer-Cartan structure equations are therefore

$$\mathbf{d}\omega^a{}_b = \omega^c{}_b{}^a{}_c + \mathbf{f}_b \mathbf{e}^a - \eta^{ac}\eta_{bd}\mathbf{f}_c\mathbf{e}^d \qquad (28)$$

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^c \omega^a{}_c + \mathbf{W}\mathbf{e}^a \qquad (29)$$

$$\mathbf{d}\mathbf{f}_a = \omega^c{}_a \mathbf{f}_c + \mathbf{f}_a \mathbf{W} \qquad (30)$$

$$\mathbf{d}\mathbf{W} = \mathbf{e}^a \mathbf{f}_a \qquad (31)$$

So far, these structure equations look the same regardless of how the group is gauged. However, there are different ways to proceed from here because there is more than one sensible subgroup. In principle, we may take the quotient of the conformal group by any subgroup, as long as that subgroup contains no normal subgroup of the conformal group. However, we certainly want the final result to permit local rotations and local dilatations.

This which restricts consideration to subgroups generated by subsets of $\{M^a{}_b, P_a, K_a, D\}$ and not, for example, collections such as $\{P_2, K_2, D\}$. Looking at the Lie algebra, we see only three rotationally and dilatationally covariant subgroups satisfying this condition, namely, those generated by one of the following three sets of generators

$$\begin{array}{c}\{M^a{}_b, P_a, D\} \\ \{M^a{}_b, K_a, D\} \\ \{M^a{}_b, D\}\end{array}$$

The first two generate isomorphic subgroups, so there are really only two independent choices, $\{M^a{}_b, K_a, D\}$ and $\{M^a{}_b, D\}$. The most natural choice is the first because it results once again in a gauge theory of a 3-dim Euclidean space. However, it leads only to a conformally flat 3-geometry with no new features. The final possibility, $\{M^a{}_b, D\}$, is called biconformal gauging. It turns out to be interesting.

Therefore, we perform the biconformal gauging, choosing the homogeneous Weyl group generated by $\{M^a{}_b, D\}$ for the local symmetry. This means that the forms $\mathbf{e}^a$ and $\mathbf{f}_a$ are independent, spanning a 6-dimensional sub-manifold of the conformal group manifold.

The solution of the structure equations (see [3]), eqs.(28-31) may be put in the form:

$$\begin{aligned}\omega^a{}_b &= (\delta^a_d \delta^c_b - \eta^{ac}\eta_{db})\, y_c \mathbf{d}x^d \\ \mathbf{W} &= -y_a \mathbf{d}x^a \\ \mathbf{e}^a &= \mathbf{d}x^a \\ \mathbf{f}_a &= \mathbf{d}y_a - \left(y_a y_b - \frac{1}{2} y^2 \eta_{ab} \right) \mathbf{d}x^b \end{aligned} \tag{32}$$

Notice that if we set $y_a = 0$, these forms reduce to

$$\begin{aligned}\omega^a{}_b &= 0 \\ \mathbf{W} &= 0 \\ \mathbf{e}^a &= \mathbf{d}x^a \\ \mathbf{f}_a &= 0\end{aligned}$$

which defines a 3-dim space Euclidean space with orthonormal basis $\mathbf{e}^a = \mathbf{d}x^a$. If, on the other hand, we hold $x^a = 0$ (or any constant), then

$$\begin{aligned}\omega^a{}_b &= 0 \\ \mathbf{W} &= 0 \\ \mathbf{e}^a &= 0 \\ \mathbf{f}_a &= \mathbf{d}y_a\end{aligned}$$

and we have a Euclidean 3-space with orthonormal basis $\mathbf{f}_a$.

We can see that $\mathbf{e}^a\mathbf{f}_a$ is a symplectic form because $\mathbf{e}^a$ and $\mathbf{f}_a$ are independent, making this 2-form non-degenerate, while the structure equation, eq.(31),

$$\mathbf{dW} = \mathbf{e}^a\mathbf{f}_a \tag{33}$$

shows that $\mathbf{e}^a\mathbf{f}_a$ is closed, $\mathbf{d}(\mathbf{e}^a\mathbf{f}_a) = \mathbf{d}^2\mathbf{W} = 0$. This is also evident from the solution, where

$$\begin{aligned}
\mathbf{e}^a\mathbf{f}_a &= \mathbf{d}x^a\left(\mathbf{d}y_a - \left(y_a y_b - \frac{1}{2}y^2\eta_{ab}\right)\mathbf{d}x^b\right) \\
&= \mathbf{d}x^a\mathbf{d}y_a
\end{aligned}$$

is in the canonical form guaranteed by the Darboux theorem ([11],[12]). Because of this symplectic form we are justified in identifying the solution as a relative of phase space.

The symplectic form allows us to define canonical brackets, analogous to Poisson brackets, which in this context we call biconformal brackets. Then the pair (x^α, y_α) satisfies the fundamental biconformal bracket relationship

$$\{x^\alpha, y_\beta\} = \delta^\alpha_\beta\ . \tag{34}$$

It is straightforward to show that a transformation is canonical if and only if it preserves this bracket.

From eq.(34) it follows that y_β is the conjugate variable to the position coordinate x^α and in mechanical units we may set $y_\alpha = \alpha_0 p_\alpha$,where p_α is momentum and y_β has units of inverse length. As discussed in Sec.3, α_0 may be any constant with the appropriate dimensions.

7.1 Single particle Hamiltonian dynamics

Since we are in a 6-dimensional symplectic space, we cannot simply write Newton's law as before. Moreover, with the interpretation as a phase space, we do not expect physical paths to be geodesics. Therefore, we postulate an action. Noting that the geometry contains a new one-form, the Weyl vector, it is reasonable to examine what paths are determined by its extremals. Therefore, we consider the action

$$S = -\int \mathbf{W}$$

Variation of S leads to the equation for a straight line. However, the results are more interesting if we start with the relativistic conformal group, $SO(4,2)$, then take an explicit Newtonian limit.

We gauge the Lorentz-conformal group, $SO(4,2)$ just as we gauged $SO(4,1)$. The Lie algebra, structure equations and the solution for the connection (eqs.27-32) are unchanged except for the range of the indices, $A, B = 0, 1, 2, 3$. In particular, the Weyl vector takes the form

$$\mathbf{W} = -y_0\mathbf{d}t - y_m\mathbf{d}x^m$$

where we identify $x^A = (t, x^a)$ with spacetime coordinates. The action now takes the form

$$\begin{aligned} S &= \int (y_0 dt + y_m dx^m) \\ &= \int \left(y_0 + y_m \frac{dx^m}{dt} \right) dt \\ &= \alpha_0 \int_c \left(p_0 + p_m \frac{dx^m}{dt} \right) dt \end{aligned}$$

Before varying S to find the equations of motion, we restrict to the Newtonian case. Specifically, we require that time, t, be universal. As a result, we cannot vary t in the action. Moreover, variation of the fundamental biconformal bracket for t implies

$$\begin{aligned} 0 &= \delta \{t, p_0\} \\ &= \{\delta t, p_0\} + \{t, \delta p_0\} \\ &= \frac{\partial (\delta p_0)}{\partial p_0}. \end{aligned} \tag{35}$$

Thus, the most general allowed variation δp_0 of p_0 depends only on the remaining coordinates, $\delta p_0 = -\delta H (y_i, x^j, t)$. Since variation may take us to any allowed value of p_0, p_0 itself is dependent on the other seven coordinates,

$$p_0 = -H (y_a, x^a, t)$$

Thus, the existence of a Hamiltonian may be viewed a consequence of the existence of universal time, and is intimately related to relativistic mechanics.

Varying the action now leads to

$$\begin{aligned} 0 &= \delta S \\ &= \alpha_0 \int_c \left(-H + p_m \frac{dx^m}{dt} \right) dt \\ &= \alpha_0 \int \left(-\frac{\partial H}{\partial x^i} \delta x^i - \frac{\partial H}{\partial p_i} \delta p_i + \delta p_i \frac{dx^i}{dt} - \frac{dp_i}{dt} \delta x^i \right) dt \end{aligned}$$

which immediately gives us Hamilton's equations for the classical paths.

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \tag{36}$$

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \tag{37}$$

We immediately recognize H as the system Hamiltonian. Notice that the arbitrary unit choice α_0 is absent from the equations of motion.

As expected, the symmetry of these equations includes local rotations and local dilatations, but in fact is larger since, as we know, local symplectic transformations preserve Hamilton's equations.

7.2 Multiparticle mechanics

Generalizing to the case of N particles, the action becomes a functional of N distinct curves, $C_n, n = 1, \ldots, N$

$$S = -\sum_{n=1}^{N} \int_{C_n} \mathbf{W} \tag{38}$$

As for the single particle case, the invariance of time constrains p_0. However, since $\mathbf{W} = -y_\alpha \mathbf{d}x^\alpha$ is to be evaluated on N different curves, there will be N distinct coordinates $x^\alpha_{(n)}$ and momenta, $p^{(n)}_\alpha$. Therefore, we have

$$\begin{aligned} 0 &= \delta \left\{ x^0_{(m)}, p^{(n)}_0 \right\} \\ &= \left\{ \delta x^0_{(m)}, p^{(n)}_0 \right\} + \left\{ x^0_{(m)}, \delta p^{(n)}_0 \right\} \\ &= \frac{\partial \left(\delta p^{(n)}_0 \right)}{\partial p^{(k)}_0} \frac{\partial x^0_{(m)}}{\partial x^0_{(k)}} \end{aligned} \tag{39}$$

Now, since time is universal in non-relativistic physics, we may set $x^0_{(m)} = t$ for all m. Therefore, $\frac{\partial x^0_{(m)}}{\partial x^0_{(k)}} = 1$ and we have

$$\frac{\partial \left(\delta p^{(n)}_0 \right)}{\partial p^{(k)}_0} = 0 \tag{40}$$

which implies that each $p^{(n)}_0$ is a function of the $6N$ spatial components only,

$$p^{(n)}_0 = -H_{(n)} \left(x^i_{(1)}, \ldots, x^i_{(N)}, p^{(1)}_i, \ldots, p^{(N)}_i \right)$$

This means that each $p^{(n)}_0$ is sufficiently general to provide a generic Hamiltonian, so that the collective Hamiltonian, defined as

$$H = \sum_{n=1}^{N} H_{(n)} \left(x^i_{(1)}, \ldots, x^i_{(N)}, p^{(1)}_i, \ldots, p^{(N)}_i \right),$$

is also obviously generic. Notice that this procedure is invertible, since we may always divide a given collective Hamiltonian into N identical parts, $H = \frac{1}{N} \sum H$, setting $H_{(n)} = H$.

Returning to the action again use the assumption of universal time, $dt_{(n)} = dt$, to write

$$\begin{aligned} S &= -\sum_{n=1}^{N} \int_{C_n} \mathbf{W} \\ &= \alpha_0 \int \sum_{n=1}^{N} \left(p^{(n)}_0 dt_{(n)} + p^{(n)}_i dx^i_{(n)} \right) \end{aligned}$$

$$= \alpha_0 \int \sum_{n=1}^{N} \left(p_0^{(n)} + p_i^{(n)} \frac{dx^i_{(n)}}{dt} \right) dt$$
$$= \alpha_0 \int_{C_n} \left(\sum_{n=1}^{N} p_i^{(n)} \frac{dx^i_{(n)}}{dt} - H \right) dt$$

wherein we recognize the usual expression for the Lagrangian in terms of the Hamiltonian,

$$L = \sum_{n=1}^{N} p_i^{(n)} \frac{dx^i_{(n)}}{dt} - H$$

Notice, however, that we have now derived both the Hamiltonian and the Lagrangian from the Weyl vector, as well as the usual Legendre transform between them.

The introduction of multiple biconformal coordinates has consequences for the biconformal structure equations as well. Though mathematically equivalent, there is a conceptual difference between the two sides of

$$\sum_{n=1}^{N} \int_{C_n} \mathbf{W} = -\alpha_0 \int \sum_{n=1}^{N} \left(p_0^n + p_i^n \frac{dx^i_n}{dt} \right) dt$$

On the left, we sum N integrals, but on the right we may interpret the sum as giving a new gauge vector,

$$\mathbf{W} = -\alpha_0 \sum_{i=1}^{N} p_\alpha^n \mathbf{d} x_n^\alpha = -\sum_{i=1}^{N} y_\alpha^n \mathbf{d} x_n^\alpha$$

With the latter interpretation the exterior derivative of $\mathbf{W}$ is

$$\mathbf{dW} = -\sum_{i=1}^{N} \mathbf{d} y_\alpha^n \mathbf{d} x_n^\alpha$$

and the structure equation

$$\mathbf{dW} = \mathbf{e}^a \mathbf{f}_a$$

must be modified to include the proper number of degrees of freedom. We therefore modify the structure equation to

$$\mathbf{dW} = \mathbf{e}^a_{(n)} \mathbf{f}_a^{(n)}$$

The remaining structure equations are satisfied by simply making the same replacement, $(\mathbf{e}^a, \mathbf{f}_a) \rightarrow \left(\mathbf{e}^a_{(n)}, \mathbf{f}_a^{(n)} \right)$. Thus we see that the introduction of multiple particles leads to multiple copies of biconformal space, in precise correspondence to the introduction of a $6N$-dim (or $8N$-dim) phase space in multiparticle Hamiltonian dynamics. These observations suggest that the symplectic structure encountered in dynamical systems has its origin in the symmetry of Newtonian measurement theory.

Finally, we note the simple relationship between the original 6-dim biconformal space and the $6N$-dim multiparticle space. Consider the cotangent space of the biconformal space at the location of any one of the N particles. This cotangent space is a copy of the flat biconformal space. If we build the direct product of these tangent spaces at the positions of all N particles, we arrive at a space which is locally isomorphic to the $6N$-dim phase space. Thus, we see that the phase space is in one to one correspondence with a subspace of the cotangent bundle of the biconformal space. The difference between the motion in phase space of a single, $6N$-dimensional vector and the motion in biconformal space of N, 6-dimensional vectors is just a matter of point of view.

One advantage of the 6-dim point of view is that we may regard biconformal spaces as fundamental in the same sense as configuration spaces, rather than derived from dynamics the way that phase spaces are. This means that in principle, dynamical systems could depend on position and momentum variables independently. While this is not so important for classical solutions, which separate into a pair of 3-dimensional submanifolds (configuration/momentum), or for relativistic solutions which similarly separate (spacetime/energy-momentum) the extended dependence on both position and momentum could yield important insights into quantum mechanics.

7.3 Is size change measurable?

While we won't systematically introduce curvature, there is one important consequence of dilatational curvature that we must examine. A full examination of the field equations for curved biconformal space ([3],[4]) shows that the dilatational curvature is proportional (but not equal) to the curl of the Weyl vector. When this curvature is nonzero, the relative sizes of physical objects may change. Specifically, suppose two initially identical objects move along paths forming the boundary to a surface. If the integral of the dilatational curvature over that surface does not vanish the two objects will no longer have identical sizes. This result is inconsistent with macroscopic physics. However, we now show that the result never occurs classically. A similar result has been shown for Weyl geometries [13].

If we fix a gauge, the change in any length dimension, l, along any path, C, is given by the integral of the Weyl vector along that path:

$$\begin{aligned} dl &= lW_A dx^A \\ l &= l_0 \exp\left(\int_C W_A dx^A\right) \end{aligned}$$

It is this integral that we want to evaluate for the special case of classical paths. Notice that this factor is gauge dependent, but if we compare two lengths which follow different paths with common endpoints, the ratio of their lengths changes in a gauge independent way:

$$\frac{l_1}{l_2} = \frac{l_{10}}{l_{20}} \exp\left(\oint_{C_1 - C_2} W_A dx^A\right)$$

This dilatation invariant result represents measurable relative size change when the exponential factor differs from unity.

We now show that such measurable size changes never occur classically. Since l_1 and l_2 must both evolve according to the classical equations of motion, the paths $C_1 = C_1\left(x_1^m, p_n^1\right)$ and $C_2 = C_2\left(x_2^m, p_n^2\right)$ are both solutions to Hamilton's equations. From the expression for the action we have

$$\begin{aligned} \int_C \mathbf{W} &= \int_C W_A dx^A \\ &= -\alpha_0 \int_{x_i}^{x_f} \left(H\left(x^n(t), p_m(t)\right) - p_k(t) \frac{dx^k(t)}{dt} \right) dt \end{aligned}$$

where $x^n(t)$ and $p_m(t)$ describe any solution to Hamilton's equations. This is just the change in Hamilton's principal function between the endpoints (see Appendix 6),

$$S\left(x_f, t_f\right) - S\left(x_i, t_i\right)$$

Any classical solution evolving from (x_i, t_i) to (x_f, t_f) gives this same result, so we always have

$$\oint_{C_1 - C_2} W_A dx^A = 0$$

and no measurable size change.

We have shown that the ratio of magnitudes of any two quantities evolving between the same initial and final points will remain constant. We can do better than this, however. Hamilton's principal function gives us a way to define a gauge in which magnitudes evolved along classical paths remain constant. Since the Weyl vector is a gauge vector, it changes inhomogeneously according to

$$\mathbf{W}' = \mathbf{W} + \mathbf{d}\phi$$

when we choose a new gauge ϕ. If we choose $\phi = S(x,t)$ then the dilatation factor for any length is

$$\begin{aligned} \exp\left(\int \mathbf{W}'\right) &= \exp\left(\int \mathbf{W} + \int \mathbf{d}S\right) \\ &= \exp\left(\int \mathbf{W} + S\right) \end{aligned}$$

But $\int \mathbf{W}$ is equal to $-S$ and the factor is unity. In this gauge, classical objects retain their magnitudes.

8 Conclusions

We have shown the following

1. The $SO(3)$ gauge theory of Newton's second law is Lagrangian mechanics
2. The $SO(4,1)$ gauge theory of Newton's second law is Hamiltonian mechanics.

These results provide a new unification of classical mechanics using the tools of gauge theory.

We note several further insights.

First, by identifying the symmetries of a theory's dynamical law from the symmetry of its measurement theory, we gain new insight into the meaning of gauge theory. Generally speaking, dynamical laws will have global symmetries while the inner products required for measurement will have local symmetries. Gauging may be viewed as enlarging the symmetry of the dynamical law to match the symmetry of measurement, thereby maintaining closer contact with what is, in fact, measurable.

Second, we strengthen our confidence and understanding of the interpretation of relativistic biconformal spaces as relatives of phase space. The fact that the same gauging applied to classical physics yields the well-known and powerful formalism of Hamiltonian dynamics suggests that the higher symmetry of biconformal gravity theories may in time lead to new insights or more powerful solution techniques.

Finally, it is possible that the 6-dimensional symplectic space of $SO(4,1)$ gauge theory represents a deep insight. Like Hamiltonian dynamics, quantum mechanics requires both position and momentum variables for its formulation – without both, the theory makes no sense. If we take this seriously, perhaps we should look closely at biconformal space as the fundamental arena for physics. Rather than regarding phase space as a convenience for calculation, perhaps there is a 6-dim (or, relativistically, 8-dim) space upon which we move and make our measurements. If this conjecture is correct, it will be interesting to see the form taken by quantum mechanics or quantum field theory when formulated on a biconformal manifold [2].

The proof of Sec. 8 is encouraging in this regard, for not only do classical paths show no dilatation, but a converse statement holds as well: non-classical paths generically do show dilatation. Since quantum systems may be regarded as sampling all paths (as in a path integral), it may be possible to regard quantum non-integrability of phases as related to non-integrable size change. There is a good reason to think that this correspondence occurs: the covering group of $SO(4,2)$ admits complex representations in which the Weyl vector is pure imaginary. This does not alter the classical results, but it changes the dilatations to phase transformations. If this is the case, then the evolution of sizes in biconformal spaces, when expressed in the usual classical variables, gives unitary evolution just as in quantum physics. The picture here is much like the familiar treatment of quantum systems as thermodynamic systems by replacing time by a complex temperature parameter, except it is now the energy-momentum vector that is replaced by a complex coordinate in a higher dimensional space. A full examination of these questions takes us too far afield to pursue here, but they are under current investigation.

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Appendices

Appendix 1: Point transformations of Newton's second law

Here we derive the point transformations leaving the second law invariant, assuming the force to transform as a vector.

Consider a general coordinate transformation in which we replace the Cartesian coordinates, x^i, as well as the time parameter, by

$$\begin{aligned} q^i &= q^i(\mathbf{x}, t) \\ \tau &= \tau(\mathbf{x}, t) \end{aligned}$$

We have four functions, each of four variables. This functions must be invertible, so we may also write

$$\begin{aligned} x^i &= x^i(\mathbf{q}, \tau) \\ t &= t(\tau) \end{aligned}$$

The limitation on covariance comes from the acceleration. First, the velocity is given by

$$\begin{aligned} v^i &= \frac{dx^i(\mathbf{q}, \tau)}{dt} \\ &= \frac{d\tau}{dt}\left(\frac{\partial x^i}{\partial q^j}\frac{dq^j}{d\tau} + \frac{\partial x^i}{\partial \tau}\right) \end{aligned}$$

where we use the usual summation convention on repeated indices, e.g.,

$$\sum_{j=1}^{3} \frac{\partial x^i}{\partial q^j}\frac{dq^j}{d\tau} = \frac{\partial x^i}{\partial q^j}\frac{dq^j}{d\tau}$$

The acceleration is

$$\begin{aligned} a^i &= \frac{dv^i(\mathbf{q}, \tau)}{dt} \\ &= \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{d\tau}{dt}\left(\frac{\partial x^i}{\partial q^j}\frac{dq^j}{d\tau} + \frac{\partial x^i}{\partial \tau}\right)\right) \\ &= \left(\frac{d\tau}{dt}\right)^2\left(\frac{\partial x^i}{\partial q^j}\frac{d^2 q^j}{d\tau^2}\right) + \frac{d^2\tau}{dt^2}\left(\frac{\partial x^i}{\partial q^j}\frac{dq^j}{d\tau} + \frac{\partial x^i}{\partial \tau}\right) \\ &\quad + \left(\frac{d\tau}{dt}\right)^2\frac{dq^k}{d\tau}\left(\frac{\partial^2 x^i}{\partial q^k \partial q^j}\frac{dq^j}{d\tau} + \frac{\partial^2 x^i}{\partial q^k \partial \tau}\right) \\ &\quad + \left(\frac{d\tau}{dt}\right)^2\left(\frac{\partial^2 x^i}{\partial \tau \partial q^j}\frac{dq^j}{d\tau} + \frac{\partial^2 x^i}{\partial \tau^2}\right) \end{aligned}$$

The first term is proportional to the acceleration of q^i, but the remaining terms are not. Since we assume that force is a vector, it changes according to:

$$F^i(\mathbf{x}, t) = \frac{\partial x^i}{\partial q^j}F^j(\mathbf{q}, \tau) \tag{41}$$

where $\frac{\partial x^i}{\partial q^j}$ is the Jacobian matrix of the coordinate transformation. Substituting into the equation of motion, we have

$$\begin{aligned}
\frac{1}{m}\frac{\partial x^i}{\partial q^j}F^j\left(\mathbf{q},\tau\right) &= \left(\frac{d\tau}{dt}\right)^2\left(\frac{\partial x^i}{\partial q^j}\frac{d^2q^j}{d\tau^2}\right) \\
&+\frac{d^2\tau}{dt^2}\left(\frac{\partial x^i}{\partial q^j}\frac{dq^j}{d\tau}+\frac{\partial x^i}{\partial\tau}\right) \\
&+\left(\frac{d\tau}{dt}\right)^2\frac{dq^k}{d\tau}\left(\frac{\partial^2x^i}{\partial q^k\partial q^j}\frac{dq^j}{d\tau}+\frac{\partial^2x^i}{\partial q^k\partial\tau}\right) \\
&+\left(\frac{d\tau}{dt}\right)^2\left(\frac{\partial^2x^i}{\partial\tau\partial q^j}\frac{dq^j}{d\tau}+\frac{\partial^2x^i}{\partial\tau^2}\right)
\end{aligned}\tag{42}$$

Newton's second law holds in the new coordinate system,

$$F^m\left(\mathbf{q},\tau\right)=m\frac{d^2q^m}{d\tau^2}$$

if and only if:

$$\begin{aligned}
1 &= \left(\frac{d\tau}{dt}\right)^2 \\
0 &= \frac{d^2\tau}{dt^2}\left(\frac{\partial x^i}{\partial q^j}\frac{dq^j}{d\tau}+\frac{\partial x^i}{\partial\tau}\right) \\
&+\left(\frac{d\tau}{dt}\right)^2\frac{dq^k}{d\tau}\left(\frac{\partial^2x^i}{\partial q^k\partial q^j}\frac{dq^j}{d\tau}+\frac{\partial^2x^i}{\partial q^k\partial\tau}\right) \\
&+\left(\frac{d\tau}{dt}\right)^2\left(\frac{\partial^2x^i}{\partial\tau\partial q^j}\frac{dq^j}{d\tau}+\frac{\partial^2x^i}{\partial\tau^2}\right)
\end{aligned}\tag{43}$$

From the first, we have

$$\tau=t+t_0$$

together with the possibility of time reversal,

$$\tau=-t+t_0$$

for the time parameter. Using this result to simplify the second (including $\frac{d^2\tau}{dt^2}=0$),

$$\begin{aligned}
0 = &\quad \frac{\partial^2x^i}{\partial q^k\partial q^j}\frac{dq^j}{d\tau}\frac{dq^k}{d\tau} \\
&+2\frac{\partial^2x^i}{\partial\tau\partial q^j}\frac{dq^j}{d\tau}+\frac{\partial^2x^i}{\partial\tau^2}
\end{aligned}\tag{44}$$

Now, since the components of the velocity, $\frac{dq^k}{d\tau}$, are independent we get three equations,

$$0 = \frac{\partial^2 x^i}{\partial q^k \partial q^j} \tag{45}$$
$$0 = \frac{\partial^2 x^i}{\partial q^k \partial \tau} \tag{46}$$
$$0 = \frac{\partial^2 x^i}{\partial \tau^2} \tag{47}$$

Integrating,

$$0 = \frac{\partial^2 x^i}{\partial \tau^2} \Rightarrow x^i = x_0^i \left(q^m\right) + v_0^i \left(q^m\right) \tau \tag{48}$$
$$0 = \frac{\partial^2 x^i}{\partial q^k \partial \tau} \Rightarrow 0 = \frac{\partial v_0^i}{\partial q^k} \Rightarrow v_0^i = const. \tag{49}$$

The remaining equation implies that the Jacobian matrix is constant,

$$\frac{\partial x^m}{\partial q^j} = \frac{\partial x_0^m}{\partial q^j} = J^m_{\ j} = const. \tag{50}$$

Integrating, the coordinates must be related by a constant, inhomogeneous, general linear transformation,

$$x^m = J^m_{\ j} q^j + v_0^i \tau + x_0^m \tag{51}$$
$$t = \tau + \tau_0 \tag{52}$$

together with a possible time reversal of t.

We get a 16-parameter family of coordinate systems: nine for the independent components of the nondegenerate 3×3 matrix J, three for the boosts v_0^i, three more for the arbitrary translation, x_0^m, and a single time translation.

Notice that the transformation includes the possibility of an arbitrary scale factor, $e^{-2\lambda} = \left|\det\left(J^m_{\ \ n}\right)\right|$.

Appendix 2: Special conformal transformations

In this Appendix, we show that special conformal transformations are $1-1$ and onto on compactified R^3.

In three dimensions, there are ten independent transformations preserving the inner product (or the line element) up to an overall factor: three rotations, three translations, one dilatation and three special conformal transformations. The first six of these are well-known for leaving ds^2 invariant – they form the Euclidean group for 3-dimensional space (or, equivalently the inhomogeneous orthogonal group, $ISO(3)$). The single dilatation is a simple rescaling. In Cartesian coordinates it is just

$$x^i = e^{\lambda} y^i$$

where λ is any constant. The special conformal transformations are actually a second kind of translation, performed in inverse coordinates, given by:

$$q^i = \frac{x^i + x^2 b^i}{1 + 2b^i x_i + b^2 x^2}$$

The inverse is given by:

$$x^i = \frac{q^i - q^2 b^i}{1 - 2q^i b_i + q^2 b^2}$$

Clearly, these transformations are not well-defined on all of R^3 because the denominator vanishes when

$$0 = 1 + 2b^i x_i + b^2 x^2$$

We may demand b^i different from zero since otherwise we have the identity map. Multiplying by b^2 we then find

$$\begin{aligned} 0 &= b^2 + 2b^2 b^i x_i + \left(b^2\right)^2 x^2 \\ &= \left(b^i + b^2 x^i\right)^2 \end{aligned}$$

Since the norm of a vector vanishes only if the vector itself vanishes we immediately have the unique result

$$x^i = -\frac{b^i}{b^2}$$

Therefore, with a one point compactification (adding a "point at infinity" analogous to the one point compactification of the complex plane), we can make the transformation one-to-one and onto. Specifically, we define an inverse y^i to every vector, x^i except the origin,

$$y^i = -\frac{x^i}{x^2}$$

then extend the manifold by defining the point at infinity to be the point with coordinates $y^i = 0$.

The general map now sends x^i to

$$q^i = \frac{x^i + x^2 b^i}{1 + 2b^i x_i + b^2 x^2}$$

except for $x^i = -\frac{b^i}{b^2}$, which is mapped to the point at infinity. The point at infinity is mapped to $\frac{b^i}{b^2}$.

Appendix 3: What is the velocity after a special conformal transformation?

Suppose a particle follows the path $\mathbf{x}(t)$ with velocity

$$\mathbf{v} = \frac{d\mathbf{x}(t)}{dt}$$

If we introduce new coordinates

$$\mathbf{y} = \frac{\mathbf{x} + x^2\mathbf{b}}{1 + 2\mathbf{x}\cdot\mathbf{b} + b^2x^2} = \beta^{-1}\left(\mathbf{x} + x^2\mathbf{b}\right)$$

where $\beta = (1 + 2\mathbf{x}\cdot\mathbf{b} + b^2x^2)$. Then

$$\mathbf{x}(t) = \frac{\mathbf{y} - y^2\mathbf{b}}{1 - 2\mathbf{y}\cdot\mathbf{b} + b^2y^2}$$

and differentiating,

$$\frac{\partial y^i}{\partial x^j} = \beta^{-1}\left(\delta^i_j + 2x_jb^i\right) - \beta^{-2}\left(x^i + x^2b^i\right)\left(2b_j + 2b^2x_j\right) \tag{53}$$

This is just as complicated as it seems. The velocity in the new coordinates is

$$\begin{aligned}\frac{dy^i}{dt} &= \beta^{-1}\left(\mathbf{v} + 2\left(\mathbf{x}\cdot\mathbf{v}\right)\mathbf{b}\right) - \beta^{-2}\left(\mathbf{x} + x^2\mathbf{b}\right)\left(2\mathbf{v}\cdot\mathbf{b} + 2b^2\left(\mathbf{x}\cdot\mathbf{v}\right)\right) \\ &= v^j\left(\beta^{-1}\left(\delta^i_j + 2x_jb^i\right) - \beta^{-2}\left(x^i + x^2b^i\right)\left(2b_j + 2b^2x_j\right)\right)\end{aligned} \tag{54}$$

The explicit form is probably the basis for Weinberg's claim [14] , that under conformal transformations "...the statement that a free particle moves at constant velocity [is] not an invariant statement...." This is clearly the case – if $v^i = \frac{dx^i}{dt}$ is constant, $\frac{dy^i}{dt}$ depends on position in a complicated way. Indeed, as shown in Sec.3, constants become position dependent as well, though there remains an invariant spectrum.

To understand the velocity transformation, note that using eq.(53) we may rewrite eq.(54) in the usual form for the transformation of a vector.

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial y^i}{\partial x^j}v^j$$

This is the reason we must introduce a derivative operator covariant with respect to special conformal transformations. The statement $v^kD_kv^i = 0$ is then a manifestly conformally covariant expression of constant velocity.

Appendix 4: The geometry of special conformal transformations

We have shown that

$$g_{ij} = \beta^{-2}\eta_{ij}$$

But notice that, if we perform such a transformation, the connection and curvature no longer vanish, but are instead given by

$$\begin{aligned}
\mathbf{e}^a &= \beta^{-1}\mathbf{d}x^a \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\omega_b^a \\
\mathbf{R}_b^a &= \mathbf{d}\omega_b^a - \omega_b^c\omega_c^a
\end{aligned}$$

The form of the curvature is given in many places (eg. [6]), but we provide the simple derivation here for completeness. From the second equation it follows that

$$\omega_b^a = -\left(\beta_{,b}\mathbf{e}^a - \eta^{ac}\eta_{bd}\beta_{,c}\mathbf{e}^d\right)$$

Then substituting into the curvature,

$$\begin{aligned}
\mathbf{R}_b^a &= \mathbf{d}\omega_b^a - \omega_b^c\omega_c^a \\
&= -\delta_d^a\beta_{,bc}\mathbf{e}^c\mathbf{e}^d + \eta^{ae}\eta_{bd}\beta_{,ec}\mathbf{e}^c\mathbf{e}^d + \delta_d^a\beta_{,c}\beta_{,b}\mathbf{e}^c\mathbf{e}^d - \eta^{ae}\eta_{bd}\beta_{,c}\beta_{,e}\mathbf{e}^c\mathbf{e}^d \\
&\quad -\delta_d^a\beta_{,b}\beta_{,c}\mathbf{e}^c\mathbf{e}^d + \delta_d^a\eta_{bc}\eta^{fe}\beta_{,f}\beta_{,e}\mathbf{e}^c\mathbf{e}^d - \eta_{bc}\eta^{af}\beta_{,d}\beta_{,f}\mathbf{e}^c\mathbf{e}^d \\
R_{bcd}^a &= \delta_c^a\beta_{,bd} - \delta_d^a\beta_{,bc} + \eta^{ae}\eta_{bd}\beta_{,ec} - \eta^{ae}\eta_{bc}\beta_{,ed} \\
&\quad + \left(\delta_d^a\eta_{bc} - \delta_c^a\eta_{bd}\right)\eta^{fe}\beta_{,f}\beta_{,e}
\end{aligned}$$

which is pure Ricci. Since the Weyl curvature tensor vanishes for conformally flat metrics, R_{bcd}^a must be constructible from the Ricci tensor alone. To see this explicitly, write the Ricci tensor and Ricci scalar,

$$\begin{aligned}
R_{bd} &= (n-2)\,\beta_{,bd} + \eta_{bd}\eta^{ce}\beta_{,ec} - (n-1)\,\eta_{bd}\eta^{fe}\beta_{,f}\beta_{,e} \\
R &= 2\,(n-1)\,\eta^{bd}\beta_{,bd} - n\,(n-1)\,\eta^{fe}\beta_{,f}\beta_{,e}
\end{aligned}$$

where

$$\begin{aligned}
\partial_a\beta &= \partial_a\left(1 - 2x\cdot b + x^2b^2\right) \\
&= -2b_a + 2b^2x_a \\
\partial_{ab}\beta &= \partial_b\left(-2b_a + 2b^2x_a\right) \\
&= 2b^2\eta_{ab}
\end{aligned}$$

so finally,

$$\begin{aligned}
R_{bd} &= 4\,(n-1)\,b^2\,(1-\beta)\,\eta_{bd} \\
R &= 4n\,(n-1)\,b^2\,(1-\beta)
\end{aligned}$$

The full curvature therefore is determined fully by the Ricci scalar:

$$\begin{aligned} R^a_{bcd} &= \left(\delta^a_c \eta_{bd} - \delta^a_d \eta_{bc}\right) 4b^2 \left(1 - \beta\right) \\ &= \frac{R}{n\left(n-1\right)} \left(\delta^a_c \eta_{bd} - \delta^a_d \eta_{bc}\right) \end{aligned}$$

where

$$R = 4n\left(n-1\right) b^2 \left(2x \cdot b - x^2 b^2\right)$$

We may also write

$$R_{ab} - \frac{1}{4} R \eta_{ab} = 0$$

Appendix 5: The Lie algebras $iso(3)$, $so(4,1)$ and $so(4,2)$

For our gauging, we require the form of the Euclidean Lie algebra $iso\,(3)$, the Euclidean-conformal Lie algebra $so\,(4,1)$ and the Lorentz-conformal Lie algebra $so\,(4,2)$. We can find all three from the general form of any pseudo-orthogonal Lie algebra. Let $\eta_{AB} = diag(1,\ldots,1,-1,\ldots,-1)$, with p positive and q negative values, be the pseudo-metric. Then the Lie algebra $o(p,q)$ is

$$\left[M^A{}_B, M^C{}_D\right] = -\frac{1}{2}\left(\delta^C_B M^A{}_D - \eta_{BD}\eta^{CE}M^A{}_E - \eta^{AC}\eta_{BE}M^E{}_D - \delta^A_D M^C{}_B\right)$$

where the generators are $M^A{}_B = \eta^{AC}M_{CB}$ and $M_{AB} = -M_{BA}$. We evaluate this for $so\,(4,2)$ then find $so\,(4,1)$ and $iso\,(3)$ as sub-algebras.

First, from among the M_{AB}, we identify the generators of Lorentz transformations, translations, special conformal transformations and dilatations. Let $A, B = 0,1,\ldots,5$ and $\alpha,\beta = 0,1,2,3$ and rotate coordinates so that the $(p,q) = (4,2)$ metric takes the form

$$\eta_{AB} = \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix}$$

Then identifying

$$\begin{aligned} \frac{1}{2}P_\alpha &= M_{\alpha 4} = -M_{4\alpha} \\ -\frac{1}{2}K_\alpha &= M_{\alpha 5} = -M_{5a} \\ D &= -2M_{45} = 2M_{54} \end{aligned}$$

we have,

$$\begin{aligned} \left[M^\alpha{}_\beta, M^\mu{}_\nu\right] &= -\frac{1}{2}\left(\delta^\mu_\beta M^\alpha{}_\nu - \eta_{\beta\nu}\eta^{\mu\sigma}M^\alpha{}_\sigma - \eta^{\alpha\mu}\eta_{\beta\sigma}M^\sigma{}_\nu - \delta^\alpha_\nu M^\mu{}_\beta\right) \\ \left[P_\alpha, M^\mu{}_\nu\right] &= -\frac{1}{2}\left(\delta^\mu_\alpha\delta^\beta_\nu - \eta_{\alpha\nu}\eta^{\mu\beta}\right)P_\beta \\ \left[K^\alpha, M^\mu{}_\nu\right] &= \frac{1}{2}\left(\delta^\alpha_\nu\delta^\mu_\beta - \eta^{\alpha\mu}\eta_{\nu\beta}\right)K^\beta \\ \left[P_\alpha, K^\mu\right] &= 2M^\mu{}_\alpha + \delta^\mu_\alpha D \\ \left[D, P_\alpha\right] &= P_\alpha \\ \left[D, K^\alpha\right] &= -K^\alpha \end{aligned}$$

with all other commutators vanishing. This is the usual form of the conformal algebra. The matrices $M^\alpha{}_\beta$ generate Lorentz transformations, the four generators P_α lead to spacetime translations, K^{α} give translations of the point at infinity (special conformal transformations), and D generates dilatations. Restricting $\alpha = (0, a) = (0, 1, 2, 3)$ to the spatial indices, we immediately recognize the $so(4,1)$ sub-algebra

$$\begin{aligned}
\left[M^{a}{}_{b}, M^{c}{}_{d}\right] &= -\frac{1}{2}\left(\delta^{c}_{b} M^{a}{}_{d} - \eta_{bd}\eta^{ce} M^{a}{}_{e} - \eta^{ac}\eta_{be} M^{e}{}_{d} - \delta^{a}_{d} M^{c}{}_{b}\right) \\
\left[P_{a}, M^{c}{}_{d}\right] &= -\frac{1}{2}\left(\delta^{c}_{a}\delta^{b}_{d} - \eta_{ad}\eta^{cb}\right) P_{b} \\
\left[K^{a}, M^{c}{}_{d}\right] &= \frac{1}{2}\left(\delta^{a}_{d}\delta^{c}_{b} - \eta^{ac}\eta_{db}\right) K^{b} \\
\left[P_{a}, K^{c}\right] &= 2M^{c}{}_{a} + \delta^{c}_{a} D \\
\left[D, P_{a}\right] &= P_{a} \\
\left[D, K^{a}\right] &= -K^{a}
\end{aligned}$$

This has the immediate $iso(3)$ subalgebra

$$\begin{aligned}
\left[M^{a}{}_{b}, M^{c}{}_{d}\right] &= -\frac{1}{2}\left(\delta^{c}_{b} M^{a}{}_{d} - \eta_{bd}\eta^{ce} M^{a}{}_{e} - \eta^{ac}\eta_{be} M^{e}{}_{d} - \delta^{a}_{d} M^{c}{}_{b}\right) \\
\left[P_{a}, M^{c}{}_{d}\right] &= -\frac{1}{2}\left(\delta^{c}_{a}\delta^{b}_{d} - \eta_{ad}\eta^{cb}\right) P_{b}
\end{aligned}$$

While these relations describe $iso(n)$ in any dimension n, in 3-dim we can simplify the algebra using the Levi-Civita tensor to write

$$\begin{aligned}
J_{a} &= -\frac{1}{2}\varepsilon_{a}{}^{bc} M_{bc} \\
M_{ab} &= -\varepsilon_{ab}{}^{c} J_{c}
\end{aligned}$$

Then we have the familiar form of $iso(3)$,

$$\begin{aligned}
\left[J_{a}, J_{b}\right] &= \varepsilon_{ab}{}^{c} J_{c} \\
\left[J_{a}, P_{b}\right] &= \varepsilon_{ab}{}^{c} P_{c} \\
\left[P_{a}, P_{b}\right] &= 0
\end{aligned}$$

Appendix 6: Hamilton's principal function

Though the existence and properties of Hamilton's principal function are well-known, we give a brief proof of its existence here because the result is central to the non-measurability of physical size change. This existence depends on the integrability of the Weyl vector along classical paths, since

$$\begin{aligned} S(x,t) &= -\alpha_0 \int_{x_0,t_0}^{x,t} \left(H\left(x^n(t), p_m(t)\right) - p_k(t) \frac{dx^k(t)}{dt} \right) dt \\ &= -\int_C W_A dx^A \\ &= -\int_C \mathbf{W} \end{aligned}$$

where $x^n(t), p_m(t)$ describe any solution to Hamilton's equations which passes through the initial and final points. In order for $S(x,t)$ to be a function, the result of this integration must be independent of which classical path is chosen. Using Stoke's theorem, the difference between any two such integrals is given by

$$\begin{aligned} \int_{C_1} \mathbf{W} - \int_{C_2} \mathbf{W} &= \oint_{C_1 - C_2} \mathbf{W} \\ &= \iint_S \mathbf{dW} \end{aligned}$$

where C_1 and C_2 are classical paths. This vanishes if and only if $\mathbf{dW} = 0$. Computing, we have

$$\begin{aligned} \mathbf{W} &= \alpha_0 \left(p_m \mathbf{d}x^m - H\mathbf{d}t\right) \\ \mathbf{dW} &= \alpha_0 \left(\mathbf{d}p_m \wedge \mathbf{d}x^m - \mathbf{d}H \wedge \mathbf{d}t\right) \\ &= \alpha_0 \left(\mathbf{d}p_m \wedge \mathbf{d}x^m - \frac{\partial H}{\partial p_m} \mathbf{d}p_m \wedge \mathbf{d}t - \frac{\partial H}{\partial x^m} \mathbf{d}x^m \wedge \mathbf{d}t \right) \\ &= \alpha_0 \left(\mathbf{d}p_m + \frac{\partial H}{\partial x^m} \mathbf{d}t \right) \wedge \left(\mathbf{d}x^m - \frac{\partial H}{\partial p_m} \mathbf{d}t \right) \\ &= 0 \end{aligned}$$

where the final result follows from Hamilton's equations. Thus, the integral of the Weyl vector is a function,

$$S(x,t) = -\int_{x_0}^{x} \mathbf{W}$$

when evaluated on classical paths.