Problems

September 14, 2018

1. Find the equations of motion by varying the following action functionals, then check whether you get the same thing from the Euler-Lagrange equation:

   (a) $S[x] = \int \exp\left(\alpha \mathbf{x}^2 + \beta \mathbf{v}^2\right) dt$ for constants α and β.

   (b) $S[x] = \int f\left(\mathbf{x}^2 \mathbf{v}^2\right) dt$ for any given function, f.

   (c) $S[x] = \frac{1}{\int \mathbf{x}\cdot\mathbf{a}dt} + \int \mathbf{x}\cdot\mathbf{a}dt$, where $\mathbf{a} = \ddot{\mathbf{x}}$. Check that your answer makes sense. Is there a Lagrangian?

2. For a particle of mass m falling radially (so it is a 1-dimensional problem) in a (Newtonian) gravitational field $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$, find the action. Show, using a scaling argument, the relationship between the period of any periodic orbit and the radius of the orbit (Kepler's law). Notice that only M can appear in the period, since m is only an overall factor. You only need to scale r, M, G, t.

3. Problem 7, Goldstein: A chain of uniform density and length L greater then the distance between its endpoints hangs between points (x_1, y_1) and $(x_{2,}y_2)$ with gravitational potential mgy. Find the shape of the chain.

4. Problem 12, Goldstein: A small hoop of radius r rolls without slipping on a large, fixed hoop of radius $R > r$. Using a Lagrange multiplier to find the force of constraint, determine at what point the small hoop loses contact with the fixed one.

5. Goldstein, Problem 14, modified: Find the equation of motion for the Lagrangian

$$L = \frac{1}{2}e^{\gamma t}\left[m\dot{\mathbf{q}}^2 - k\mathbf{q}^2\right]$$

   where $\mathbf{q}$ lies in the xy plane. What quantity is conserved? Describe the system by finding and recognizing its equation of motion. Now rewrite the Lagrangian in terms of s where $\mathbf{s} = e^{\gamma t}\mathbf{q}$. Show the equation of motion for s has the opposite sign for γ. Now what is the conserved quantity? Is it the same or has $\gamma \to -\gamma$?

6. Using Lagrange multipliers, find the motion of a block of mass m sliding down a frictionless plane (the sloping side of a wedge-shaped block of mass M initially at rest on a frictionless table) inclined at angle θ. Find the motion of the block and wedge.