Problem Set I.a

August 30, 2018

Problem Set 1: Due Wednesday, September 5

You may work all problems using Newton's second law.

1. The moment of inertia of a body with density ρ about any given axis is given by summing up $\rho r^2 d^3x$ for each infinitesimal volume d^3x of the body, where r is the perpendicular distance from the axis to the volume element,

   $$I = \int\limits_{body} \rho r^2 d^3x$$

   From this, show that the moment of inertia of a flat circular disk of radius R, thickness L and mass M is $\frac{1}{2}MR^2$.

2. Atwood's machine with massive rope. Repeat the problem, but let the rope be of length L and have total uniform mass, m. Find the acceleration a. A rope passes over a uniform pulley of mass, M, and radius, R. Masses m_1 and m_2, with $m_1 < m_2$, are attached to the ends of the rope. The system rotates toward m_2 with m_1 rising and m_2 falling with the same acceleration a. Take the direction of motion as positive for all forces and velocities. Find the acceleration a.

3. Use conservation of energy to find the escape velocity from Earth, where the potential is $V(r) = -\frac{GM_E}{r}$.

4. Rocket problem. A rocket is projected vertically upward, ejecting fuel with velocity u relative to the rocket. Let the total mass of the rocket be M and the total mass of fuel be m. Treating the gravitational force as nearly constant, $-mg\mathbf{k}$, find the speed of the rocket as a function of the mass of the fuel, $v(m)$. What fraction, $\frac{m}{M}$, of the total mass must be fuel to reach escape velocity?

5. We have proved that the angular momentum,

   $$\mathbf{L} = (\mathbf{r}(t) - \mathbf{R}) \times \mathbf{p}(t)$$

   of a single particle about a fixed position, $\mathbf{R}$, is related to the torque,

   $$\mathbf{N} = (\mathbf{r}(t) - \mathbf{R}) \times \mathbf{F}(t)$$

   by the equation

   $$\mathbf{N} = \frac{d\mathbf{L}}{dt}$$

   Now consider an *isolated system of particles* (this means there are no external forces, only the forces between the particles of the system) with positions $\mathbf{r}_i$ and momenta $\mathbf{p}_i$. Let the force *on* the i^{th} particle *by* the j^{th} particle be $\mathbf{F}_{ij}$. Assume Newton's 3^{rd} Law holds, so that

   $$\begin{aligned} \mathbf{F}_{ji} &= -\mathbf{F}_{ij} \\ \mathbf{F}_{ii} &= 0 \end{aligned}$$

Assume that the force between any two of the particles is a central force, i.e., it lies along the vector, $\mathbf{r}_i(t) - \mathbf{r}_j(t)$, between the two particles. The angular momentum of the i^{th} particle is then $\mathbf{L}_i = (\mathbf{r}_i(t) - \mathbf{R}) \times \mathbf{p}_i(t)$ with the torque on the i^{th} particle produced by the j^{th} given by $\mathbf{N}_{ij} = (\mathbf{r}_i(t) - \mathbf{R}) \times \mathbf{F}_{ij}(t)$ so the total torque on the i^{th} particle is

$$\mathbf{N}_i = \sum_{j=1}^{N} (\mathbf{r}_i(t) - \mathbf{R}) \times \mathbf{F}_{ij}(t)$$

therefore satisfies,

$$\mathbf{N}_i = \frac{d\mathbf{L}_i}{dt}$$

Show that the total angular momentum of the system,

$$\mathbf{L}_{tot} = \sum_{i=1}^{N} \mathbf{L}_i$$

is conserved.