Central Forces

October 14, 2018

1 Angular momentum

Because central forces are rotationally symmetric, we begin our discussion by finding the Noether conservation laws corresponding to rotations. Returning to Noether's theorem recall that the conserved quantity for an infinitesimal symmetry $\delta x^i = \varepsilon^i(x)$ is

$$I = \frac{\partial L(x(\lambda))}{\partial \dot{x}^i} \varepsilon^i(x)$$

Consider a 3-dimensional system with a rotationally symmetric Lagrangian. In Cartesian coordinates we may write

$$\begin{aligned} L &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - V(r) \\ &= \frac{1}{2} m \dot{\mathbf{x}}^2 - V(r) \end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. We easily see that the gradient with respect to the velocity, $[\boldsymbol{\nabla}_{\dot{\mathbf{x}}} L]_i = \frac{\partial L(x(\lambda))}{\partial \dot{x}^i}$, is the linear momentum,

$$\boldsymbol{\nabla}_{\dot{\mathbf{x}}} L = m \dot{\mathbf{x}} = \mathbf{p}$$

and need only the effect of an infinitesimal rotation.

To find those infinitesimal transformations that preserve L must preserve both

$$\begin{aligned} r^2 &= \mathbf{x} \cdot \mathbf{x} \\ v^2 &= \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} \end{aligned}$$

Consider an infinitesimal transformation,

$$\begin{aligned} \tilde{\mathbf{x}} &= \mathbf{x} + \delta \mathbf{x} \\ \dot{\tilde{\mathbf{x}}} &= \dot{\mathbf{x}} + \delta \dot{\mathbf{x}} \end{aligned}$$

Substituting, we require $\tilde{r}^2 = r^2$,

$$\begin{aligned} \tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}} &= \mathbf{x} \cdot \mathbf{x} \\ (\mathbf{x} + \delta \mathbf{x}) \cdot (\mathbf{x} + \delta \mathbf{x}) &= \mathbf{x} \cdot \mathbf{x} \\ \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \delta \mathbf{x} + \delta \mathbf{x} \cdot \mathbf{x} + \delta \mathbf{x} \cdot \delta \mathbf{x} &= \mathbf{x} \cdot \mathbf{x} \end{aligned}$$

Cancelling the common term and dropping the negligible quadratic term, and using the symmetry of the dot product, $\mathbf{x} \cdot \delta \mathbf{x} = \delta \mathbf{x} \cdot \mathbf{x}$,

$$\mathbf{x} \cdot \delta \mathbf{x} = 0$$

The most general vector perpendicular to $\mathbf{x}$ is a general cross product with $\mathbf{x}$, so we may have

$$\delta \mathbf{x} = \mathbf{x} \times \mathbf{b}$$

for any constant vector $\mathbf{b}$.

Now, returning to the conservation law,

$$\begin{aligned} I &= \frac{\partial L\left(x\left(\lambda\right)\right)}{\partial \dot{x}^i}\varepsilon^i\left(x\right) \\ &= \mathbf{p}\cdot\left(\mathbf{x}\times\mathbf{b}\right) \end{aligned}$$

and using the cyclic property of the triple product, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$, we have

$$I = \mathbf{b}\cdot(\mathbf{x}\times\mathbf{p})$$

Since $\mathbf{b}$ is an arbitrary constant vector, this gives us three conservation laws. We define the angular momentum vector,

$$\mathbf{L} \equiv \mathbf{x}\times\mathbf{p} \tag{1}$$

Finally, since $\delta\mathbf{x} = \mathbf{x}\times\mathbf{b}$, we have $\delta\dot{\mathbf{x}} = \dot{\mathbf{x}}\times\mathbf{b}$ and therefore $\dot{\mathbf{x}}\cdot\delta\dot{\mathbf{x}} = 0$ as well. We conclude that systems with full rotational symmetry preserve angular momentum. As seen in our discussion of Noether's theorem, rotational symmetry in any plane will lead to conservation of the corresponding component of angular momentum.

2 The general 2-body problem

We now study the motion of two point masses moving in an arbitrary central potential, that is, a potential depending only on the separation of the two bodies. The resultant force is along the vector joining the two bodies. Fixing a coordinate system, let the positions of the two masses, M and m, be $\mathbf{X}$ and $\mathbf{x}$, respectively. Then the separation of the two bodies is

$$r = |(\mathbf{X}-\mathbf{x})| = +\sqrt{(\mathbf{X}-\mathbf{x})^2}$$

and we may write the potential as $V(r)$.

2.1 Center of mass coordinates

The Cartesian form of the action,

$$S = \int_{t_1=0}^{t_2=t} \frac{1}{2}M\dot{\mathbf{X}}^2 + \frac{1}{2}m\dot{\mathbf{x}}^2 - V(r)$$

is dependent upon six coordinates. We make a simplification by introducing new coordinates, the center of mass position

$$\mathbf{R} \equiv \frac{M\mathbf{X}+m\mathbf{x}}{M+m} \tag{2}$$

and the separation vector, $\mathbf{r}\equiv\mathbf{x}-\mathbf{X}$ from M to m. In terms of these we may solve for $\mathbf{X}$ and $\mathbf{x}$:

$$\begin{aligned} (M+m)\,\mathbf{R} &= M\mathbf{X}+m\mathbf{x} \\ &= M\mathbf{X}+m\left(\mathbf{r}+\mathbf{X}\right) \\ &= (M+m)\,\mathbf{X}+m\mathbf{r} \end{aligned}$$

and dividing by the total mass,

$$\mathbf{X} = \mathbf{R} - \frac{m\mathbf{r}}{M+m}$$

Now substitute to find $\mathbf{x}$,

$$\begin{aligned}
\mathbf{x} &= \mathbf{r} + \mathbf{X} \\
&= \mathbf{r} + \mathbf{R} - \frac{m\mathbf{r}}{M+m} \\
&= \mathbf{R} + \frac{M}{M+m}\mathbf{r}
\end{aligned}$$

In terms of these, the velocities become

$$\begin{aligned}
\dot{\mathbf{X}} &= \dot{\mathbf{R}} - \frac{m\dot{\mathbf{r}}}{M+m} && (3) \\
\dot{\mathbf{x}} &= \dot{\mathbf{R}} + \frac{M\dot{\mathbf{r}}}{M+m} && (4)
\end{aligned}$$

so the kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}M\dot{\mathbf{X}}^2 + \frac{1}{2}m\dot{\mathbf{x}}^2 \\
&= \frac{1}{2}M\left(\dot{\mathbf{R}}^2 - \frac{2m\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}}{M+m} + \frac{m^2\dot{\mathbf{r}}^2}{(M+m)^2}\right) + \frac{1}{2}m\left(\dot{\mathbf{R}}^2 + \frac{2M\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}}{M+m} + \frac{M^2\dot{\mathbf{r}}^2}{(M+m)^2}\right) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 - \cancel{\frac{Mm}{M+m}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}} + \frac{1}{2}\frac{Mm^2}{(M+m)^2}\dot{\mathbf{r}}^2 + \frac{1}{2}m\dot{\mathbf{R}}^2 + \cancel{\frac{mM}{M+m}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}} + \frac{1}{2}\frac{mM^2}{(M+m)^2}\dot{\mathbf{r}}^2 \\
&= \frac{1}{2}(M+m)\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{mM}{M+m}\dot{\mathbf{r}}^2
\end{aligned}$$

We define the *total mass* and the *reduced mass*,

$$\begin{aligned}
M_0 &\equiv M+m \\
\mu &\equiv \frac{mM}{M+m}
\end{aligned}$$

and the action becomes

$$\begin{aligned}
S &= \int_0^t \left(\frac{1}{2}M_0\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r)\right)dt \\
&= \int_0^t \frac{1}{2}M_0\dot{\mathbf{R}}^2 dt + \int_0^t \left(\frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r)\right)dt
\end{aligned}$$

The action has separated into two decoupled terms, S_R and S_r with a total of twelve degrees of freedom, $\mathbf{R}, \mathbf{r}, \dot{\mathbf{R}}, \dot{\mathbf{r}}$.

Note that we may write the positions and velocities in any coordinate system, including Cartesian,

$$\begin{aligned}
\dot{\mathbf{R}}^2 &= \dot{R}_x^2 + \dot{R}_y^2 + \dot{R}_z^2 \\
\dot{\mathbf{r}}^2 &= \dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2
\end{aligned}$$

and spherical,

$$\begin{aligned}
\dot{\mathbf{R}}^2 &= \dot{R}^2 + R^2\dot{\Theta}^2 + R^2\sin^2\Theta\dot{\Phi}^2 \\
\dot{\mathbf{r}}^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2
\end{aligned}$$

We reserve this choice until we consider conserved quantities.

3 Conserved quantities

We consider consequences of Noether's theorem for the two body problem.

First, we notice that all components of the center of mass are cyclic, $\frac{\partial L}{\partial R^i} = 0$, so the conjugate linear momentum,

$$\mathbf{P} = M_0 \dot{\mathbf{R}}$$

is conserved. Writing $\dot{\mathbf{R}} = \frac{\mathbf{P}}{M_0}$ and integrating, the motion of the center of mass proceeds at constant velocity,

$$\mathbf{R} = \mathbf{R}_0 + \frac{\mathbf{P}}{M_0} t$$

This is the complete solution for six of the twelve degrees of freedom.

Now we are left with the reduced action,

$$S_r = \int_0^t \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r) \right) dt$$

describing a single particle of mass μ at position $\mathbf{r}$, moving in a central potential. Because S_r is spherically symmetric, we immediately know that the angular momentum vector,

$$\mathbf{L} = \mathbf{r} \times \mu \dot{\mathbf{r}}$$

is conserved.

Finally, L contains no explicit time dependence, so we have conserved energy,

$$\begin{aligned} E &= \frac{\partial L}{\partial \dot{r}_i} \dot{r}_i - L \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r) \right) \\ &= \mu \dot{\mathbf{r}}^2 - \left(\frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r) \right) \\ &= \frac{1}{2} \mu \dot{\mathbf{r}}^2 + V(r) \end{aligned}$$

We use the conservation of angular momentum to inform our choice of coordinates, then, with the resulting simplifications, are able to use the energy to reduce the problem to quadratures.

4 The equation of motion

We use two of the conserved angular momenta immediately. The constancy of $\mathbf{L}$ means that the position $\mathbf{r}$ and reduced momentum $\mu \dot{\mathbf{r}}$ always lie in the same plane. To see this, choose initial coordinates the angular momentum lies along the z-axis, $\mathbf{L} = L\mathbf{k}$, and must remain there. Then,

$$\begin{aligned} 0 &= \mathbf{k} \times \mathbf{L} \\ &= \mathbf{k} \times (\mathbf{r} \times \mu \dot{\mathbf{r}}) \\ &= \mathbf{r} (\mathbf{k} \cdot \mu \dot{\mathbf{r}}) - \mu \dot{\mathbf{r}} (\mathbf{k} \cdot \mathbf{r}) \end{aligned}$$

With the exception of vanishing angular momentum, which requires $\mathbf{r}$ and $\dot{\mathbf{r}}$ parallel, $\mathbf{r}$ and $\dot{\mathbf{r}}$ must lie in distinct directions. it follows that

$$\begin{aligned} \mathbf{k} \cdot \mathbf{r} &= 0 \\ \mathbf{k} \cdot \dot{\mathbf{r}} &= 0 \end{aligned}$$

and the motion lies in the xy plane at all times. Since this argument depends only on the direction of $\mathbf{L}$, the magnitude still contains useful information about the motion.

Given the orbital character of generic motion, we choose spherical coordinates. With the angular momentum along the z-axis, the motion is entirely in the xy plane, characterized by $\theta = \frac{\pi}{2}$, and this angle cannot change so $\dot{\theta} = 0, \sin\theta = 0$. Writing the action in these coordinates then gives

$$\begin{aligned} S_r &= \int_0^t \left(\frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r)\right) dt \\ &= \int_0^t \left(\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) - V(r)\right) dt \\ &= \int_0^t \left(\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)\right) dt \end{aligned}$$

We are left with only two coordinates, with φ cyclic. The conserved angluar momentum is

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mu\dot{\mathbf{r}} \\ &= \mathbf{r} \times \mu\left(\dot{r}\hat{\mathbf{r}} + r\dot{\varphi}\hat{\boldsymbol{\varphi}}\right) \\ &= \mu r^2\dot{\varphi}\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} \\ &= \mu r^2\dot{\varphi}\hat{\mathbf{k}} \end{aligned}$$

so the direction is correct and we have the conserved magnitude,

$$L = \mu r^2\dot{\varphi}$$

The constancy of the magnitude also follows directly from S_r by recognizing that φ is cyclic.

The sole remaining equation of motion, arising from varying r is

$$-\mu\ddot{r} + \mu r\dot{\varphi}^2 - \frac{\partial V}{\partial r} = 0$$

Substituting $\dot{\varphi} = \frac{L}{\mu r^2}$ we have a single, ordinary differential equation,

$$\mu\ddot{r} - \frac{L^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0 \tag{5}$$

5 Solving the equation of motion

We have shown that the action for any two body system acted on by a central force may be written as

$$S = \int_0^t \left(\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)\right) dt$$

where $\mu = \frac{mM}{M+m}$ is the reduced mass and $L = \mu r^2\dot{\varphi}$ the conserved angular momentum.

The equation of motion was found to be

$$\mu\ddot{r} - \frac{L^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0$$

but we work instead with the conserved energy,

$$\begin{aligned} E &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) \end{aligned}$$

Notice that for bound states we may have $E < 0$. The energy is fixed by its initial value. Taking $\varphi = 0$ when $r = r_{max}$ for a bounded orbit at $t = 0$,

$$E = \frac{L^2}{2\mu r_{max}^2} + V(r_{max}) \quad (6)$$

$$L = \mu r_{max}^2 \dot{\varphi}(0) \quad (7)$$

We might equally well take $\varphi = 0$ at r_{min}.

By writing the angular part of the kinetic energy in terms of r, the energy has the same form as that of a single particle of mass μ in an *effective potential*

$$V_{eff} = V(r) + \frac{L^2}{2\mu r^2}$$

A great deal about the qualitative properties of solutions can be found by studying the effective potential. The angular momentum term becomes unboundedly large at small r, creating a repulsive core to the effective potential for all but extremely attractive potentials (i.e., with $V(r) < -\frac{L^2}{2\mu r^2}$ at small r). Together with an attractive potential, there can be a minimum to V_{eff} with oscillatory bound states. We will see this in detail for the inverse-square law potential.

When the potential diverges at infinity, as for the simple harmonic potential, $V = \frac{1}{2}kx^2$, all states are bound and the only finite energy solutions are orbits. If there is an upper bound to the potential, as for inverse square laws of Newtonian gravity or the Coulomb potential, $V = -\frac{\alpha}{r}$, then it is generally convenient to take the upper bound to be zero energy. Then negative energy states are bound orbits while positive energy states have at most one turning point and escape to infinity. For bound states with $V(r) < -\frac{L^2}{2\mu r^2}$ at small r, there will be two turning points, r_{min} and r_{max} with the motion oscillating between these two values. The kinetic energy at the turning points is purely tangential, while $\frac{1}{2}\mu\dot{r}^2$ reaches a maximum at the deepest part of the potential well.

For unbound orbits, the particle escapes the center of force. There may be a single turning point, $s = r_{min}$, called the *impact parameter*. If the initial motion is toward the center of attraction, the particle reaches the turning point then moves off to infinity. Because of the time reversibility of the second law, the complete particle path (that is, for $t \in (-\infty, +\infty)$) is symmetric about the point of closest approach. At this point of closest approach, $\dot{r} = 0$, so the angular momentum and energy are given by

$$\begin{aligned} L &= \mu s^2 \dot{\varphi} \\ E &= \frac{1}{2}\mu s^2 \dot{\varphi}^2 + V(s) \\ &= \frac{L^2}{2\mu s^2} + V(s) \end{aligned}$$

so the impact parameter determines the entire motion. The full path of motion begins along a line which would miss the center of force, bending toward (for an attractive force) or away (for a repulsive force) from the center of force. It eventually reaches the turning point and begins to move off in direction altered by an angle 2Ψ. The principle problem of such *scattering* problems is to find, for a uniform collection incident particles, the dependence of $\Psi(s)$ of the scatterning angle on the impact parameter.

We treat bound state problems in detail in subsequent Sections, with a discussion of scattering in Section (11).

5.1 Solving directly

Solving for the radial velocity,

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} - V(r)\right)}$$

Using

$$\begin{aligned}\frac{dr}{d\varphi} &= \frac{dr}{dt}\frac{dt}{d\varphi} \\ &= \frac{\dot{r}}{\dot{\varphi}} \\ &= \frac{\mu r^2 \dot{r}}{L}\end{aligned}$$

we transform to an equation for $r(\varphi)$,

$$\begin{aligned}\frac{dr}{d\varphi} &= \frac{\mu r^2}{L}\sqrt{\frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} - V(r)\right)} \\ &= \frac{\sqrt{2\mu}}{L} r^2 \sqrt{E - \frac{L^2}{2\mu r^2} - V(r)}\end{aligned}$$

or, integrating,

$$\begin{aligned}\frac{\sqrt{2\mu}}{L}\int_0^{\varphi} d\varphi &= \int_{r_{max}}^{r} \frac{dr}{r^2\sqrt{E - \frac{L^2}{2\mu r^2} - V(r)}} \\ \frac{\sqrt{2\mu}}{L}\varphi &= \int_{r_{max}}^{r} \frac{dr}{r^2\sqrt{E - \frac{L^2}{2\mu r^2} - V(r)}}\end{aligned}$$

This often takes a simpler form if we replace $r = \frac{1}{u}, dr = -\frac{1}{u^2}du$,

$$\begin{aligned}\frac{\sqrt{2\mu}}{L}\varphi &= \int_{r_{max}}^{r} \frac{-\frac{1}{u^2}du}{\frac{1}{u^2}\sqrt{E - \frac{L^2}{2\mu}u^2 - V\left(\frac{1}{u}\right)}} \\ &= -\int_{r_{max}}^{r} \frac{du}{\sqrt{E - \frac{L^2}{2\mu}u^2 - V\left(\frac{1}{u}\right)}} \qquad (8)\end{aligned}$$

6 Inverse square law: gravitation

Let the potential be given by an inverse square law

$$V(r) = -\frac{\alpha}{r} = -\alpha u$$

This includes Newton's law of universal gravitation, by setting $\alpha = GMm$.g.

6.1 Bound orbits

For bound orbits, $E < 0$, let the motion lie in the xy plane and orient the axes so that r_{max} lies on the positive x-axis. Taking $\varphi = 0$ when $r = r_{max}$, we have $\dot{r}(0) = 0$. Then from the conservation of angular momentum we have $\dot{\varphi}(0) = \frac{L}{\mu r_{max}^2}$. Then

$$\begin{aligned}\frac{\sqrt{2\mu}}{L}\varphi &= -\int_{r_{max}}^{r} \frac{du}{\sqrt{E - \frac{L^2}{2\mu}u^2 + \alpha u}} \\ \frac{\sqrt{2\alpha\mu}}{L}\varphi &= -\int_{r_{max}}^{r} \frac{du}{\sqrt{u - \frac{L^2}{2\alpha\mu}u^2 + \frac{E}{\alpha}}}\end{aligned}$$

Define

$$\begin{aligned}\varepsilon &= -\frac{E}{\alpha} > 0 \\ l^2 &= \frac{L^2}{2\alpha\mu}\end{aligned} \qquad (9)$$

Notice that for bound states, *smaller* ε corresponds to *higher* energy. With these definitions,

$$-\frac{\varphi}{l} = \int_{r_{max}}^{r} \frac{du}{\sqrt{u - l^2u^2 - \varepsilon}}$$

We complete the square under the radical,

$$\begin{aligned}-l^2u^2 + u - \varepsilon &= -\left(l^2u^2 - u + \varepsilon\right) \\ &= -\left(\left(lu - \frac{1}{2l}\right)^2 - \frac{1}{4l^2} + \varepsilon\right) \\ &= \left(\frac{1}{4l^2} - \varepsilon\right) - \left(lu - \frac{1}{2l}\right)^2\end{aligned}$$

so that

$$-\varphi = \int_{r_{max}}^{r} \frac{ldu}{\sqrt{\left(\frac{1}{4l^2} - \varepsilon\right) - \left(lu - \frac{1}{2l}\right)^2}}$$

Replacing $\zeta = lu - \frac{1}{2l}$ and setting

$$c^2 \equiv \frac{1}{4l^2} - \varepsilon$$

we have

$$-\varphi = \int_{r_{max}}^{r} \frac{d\zeta}{\sqrt{c^2 - \zeta^2}}$$

Now the simple trigonometric substitution, $\zeta = c\sin\theta$, gives

$$\begin{aligned}-\varphi &= \int_{r_{max}}^{r} \frac{c\cos\theta d\theta}{\sqrt{c^2 - c^2\sin^2\theta}} \\ &= \int_{r_{max}}^{r} d\theta \\ &= \theta\big|_{r_{max}}^{r} \\ &= \sin^{-1}\frac{\zeta}{c}\bigg|_{r_{max}}^{r} \\ &= \sin^{-1}\frac{\zeta(r)}{c} - \varphi_0\end{aligned}$$

where $\varphi_0 \equiv \sin^{-1}\left(\frac{1}{c}\left(\frac{l}{r_{max}} - \frac{1}{2l}\right)\right)$. Now solving for r,

$$\begin{aligned}c\sin(\varphi - \varphi_0) &= \frac{1}{2l} - \frac{l}{r} \\ \frac{1}{r} &= \frac{1}{2l^2} - \frac{c}{l}\sin(\varphi - \varphi_0)\end{aligned}$$

Finally, inverting to give r,

$$\begin{aligned} r &= \frac{1}{\frac{1}{2l^2} - \frac{c}{l}\sin(\varphi - \varphi_0)} \\ r &= \frac{2l^2}{1 - 2lc\sin(\varphi - \varphi_0)} \end{aligned}$$

6.1.1 The standard ellipse

In addition to finding the expression for $r(\varphi)$ in terms of initial conditions, it is helpful to also express the solution in terms of the standard parameterization of an ellipse. For the parts of an ellipse, see **Parts of an ellipse**. The standard equation for an ellipse is

$$r = \frac{p}{1 + \epsilon\cos\varphi} \tag{10}$$

The height of the ellipse at $x = 0$ is found by setting $\varphi = \frac{\pi}{2}$. This is called the *semi latus rectum*, p. The coefficient of the cosine in the denominator is the *eccentricity*, ϵ.

Now fix the values of φ_0 and c. Clearly, r is maximum when $\sin(\varphi - \varphi_0) = +1$. Since we have defined φ so that this occurs when $\varphi = 0$, we have

$$\sin(-\varphi_0) = 1$$

and therefore, $\varphi_0 = -\frac{\pi}{2}$. Using

$$\begin{aligned} -\sin(\varphi - \varphi_0) &= \sin\left(\varphi + \frac{\pi}{2}\right) \\ &= \cos\varphi \end{aligned}$$

the orbit takes the form

$$r = \frac{1}{\frac{1}{2l^2} - \frac{c}{l}\sin(\varphi - \varphi_0)} = \frac{2l^2}{1 - 2lc\sin(\varphi - \varphi_0)}$$

$$r = \frac{2l^2}{1 + 2lc\cos\varphi} \tag{11}$$

Since φ_0 was originally defined as $\varphi_0 \equiv \sin^{-1}\left(\frac{1}{c}\left(\frac{l}{r_{max}} - \frac{1}{2l}\right)\right)$, we may now find c,

$$\begin{aligned} \sin\varphi_0 &\equiv \frac{1}{c}\left(\frac{l}{r_{max}} - \frac{1}{2l}\right) \\ \sin\frac{\pi}{2} &\equiv \frac{1}{c}\left(\frac{l}{r_{max}} - \frac{1}{2l}\right) \\ c &\equiv \frac{l}{r_{max}} - \frac{1}{2l} \end{aligned}$$

Comparing Eqs.(10) and (11), immediately gives expressions for the semi-latus rectum and the eccentricity,

$$p = \frac{2l^2}{1 + 2lc\cos\frac{\pi}{2}} = 2l^2$$

and

$$\begin{aligned} \epsilon &= 2lc \\ &= \frac{p}{r_{max}} - 1 \end{aligned}$$

Other parts of the standard ellipse are now easy to find in terms of p and ϵ. Eq.(10) easily gives the the maximum and minimum values of r as

$$\begin{aligned} r_{max} &= \frac{p}{1-\epsilon} \\ r_{min} &= \frac{p}{1+\epsilon} \end{aligned}$$

The length of the long axis of the ellipse is given by the sum of these so that the *semi-major axis*, a is given by

$$\begin{aligned} a &= \frac{1}{2}\left(r_{max}+r_{min}\right) \\ &= \frac{1}{2}\left(\frac{p}{1-\epsilon}+\frac{p}{1+\epsilon}\right) \\ &= \frac{1}{2}\left(\frac{p\left(1+\epsilon\right)+p\left(1-\epsilon\right)}{1-\epsilon^2}\right) \end{aligned}$$

and therefore,

$$a = \frac{p}{1-\epsilon^2} \tag{12}$$

The distance from one focus to center of the ellipse is called the *linear eccentricity.* It is given by either

$$\begin{aligned} a - r_{min} &= \frac{p}{1-\epsilon^2} - \frac{p}{1+\epsilon} \\ &= \frac{p\epsilon}{1-\epsilon^2} \\ &= a\epsilon \end{aligned}$$

or

$$r_{max} - a = \frac{p}{1-\epsilon} - \frac{p}{1-\epsilon^2} = a\epsilon$$

For any φ, the height y of the ellipse above the x-axis is

$$\begin{aligned} y &= r\sin\varphi \\ &= \frac{p\sin\varphi}{1+\epsilon\cos\varphi} \end{aligned}$$

and the *semi-minor axis*, $b = y_{max}$ is found by finding the maximum value of this expression. Setting $\frac{dy}{d\varphi} = 0$,

$$\begin{aligned} 0 &= \frac{dy}{d\varphi} \\ &= \frac{p\cos\varphi_b}{1+\epsilon\cos\varphi_b} - \frac{-p\epsilon\sin^2\varphi_b}{\left(1+\epsilon\cos\varphi_b\right)^2} \end{aligned}$$

Cancelling common terms, this leaves

$$\begin{aligned} 0 &= \cos\varphi_b\left(1+\epsilon\cos\varphi_b\right)+\epsilon\sin^2\varphi_b \\ &= \cos\varphi_b+\epsilon \end{aligned}$$

so that

$$\varphi_b = -\cos^{-1}\epsilon$$

Substituting to find b,

$$\begin{aligned} b &= y_{max} \\ &= \frac{p\sin\varphi_b}{1+\epsilon\cos\varphi_b} \\ &= \frac{p\sqrt{1-\cos^2\varphi_b}}{1+\epsilon\cos\varphi_b} \\ &= \frac{p\sqrt{1-\epsilon^2}}{1-\epsilon^2} \end{aligned}$$

so that the semi-minor axis is

$$b = \frac{p}{\sqrt{1-\epsilon^2}} \tag{13}$$

The semi-minor and semi-major axes are related by

$$b = a\sqrt{1-\epsilon^2}$$

6.1.2 Fixing the constants: Semi-latus rectum and eccentricity in terms of the energy and angular momentum

In terms of the original variables, we have

$$\begin{aligned} \varepsilon &= -\tfrac{E}{GMm} = -\frac{E}{\alpha} > 0 \\ l^2 &= \tfrac{L^2}{2GM\mu} = \frac{\mu L^2}{2\alpha m} \end{aligned}$$

but we still need to express p and ϵ in terms of ε and l. Starting with the solution in standard form, Eq.(10), we differentiate to find $\dot{r}$,

$$\begin{aligned} \dot{r} &= \frac{p\epsilon\dot{\varphi}\sin\varphi}{(1+\epsilon\cos\varphi)^2} \\ &= \frac{\epsilon r^2\dot{\varphi}\sin\varphi}{p} \end{aligned}$$

allowing us to compute the energy,

$$\begin{aligned} E &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{\alpha}{r} \\ &= \frac{1}{2}\mu\left(\frac{\epsilon^2 r^4\dot{\varphi}^2\sin^2\varphi}{p^2} + \frac{(1+\epsilon\cos\varphi)^2}{p^2}r^4\dot{\varphi}^2\right) - \frac{\alpha}{r} \\ &= \frac{1}{2}\mu r^4\dot{\varphi}^2\left(\frac{\epsilon^2\sin^2\varphi}{p^2} + \frac{(1+\epsilon\cos\varphi)^2}{p^2}\right) - \frac{\alpha}{p}(1+\epsilon\cos\varphi) \end{aligned}$$

Setting $E = -\alpha\varepsilon$ and replacing $\mu r^4\dot{\varphi}^2 = \frac{L^2}{\mu}$, this becomes

$$-\alpha\varepsilon = \frac{L^2}{2\mu p^2}\left(\epsilon^2 + 1 + 2\epsilon\cos\varphi\right) - \frac{\alpha}{p}(1+\epsilon\cos\varphi)$$

The leading coefficient reduces to

$$\frac{L^2}{2\mu p^2} = \frac{2\alpha\mu l^2}{2\mu p^2} = \frac{2\alpha l^2}{2p^2} = \frac{\alpha}{2p}$$

so that, cancelling the overall α,

$$
\begin{aligned}
-\varepsilon &= \frac{1}{2p}\left(\epsilon^2 + 1 + 2\epsilon\cos\varphi\right) - \frac{1}{p}\left(1 + \epsilon\cos\varphi\right) \\
-\varepsilon &= \frac{1}{2p}\left(\epsilon^2 + 1 + 2\epsilon\cos\varphi - 2 - 2\epsilon\cos\varphi\right) \\
-\varepsilon &= \frac{1}{2p}\left(\epsilon^2 - 1\right)
\end{aligned}
$$

and solving for the eccentricity,

$$\epsilon = \sqrt{1 - 2p\varepsilon} \tag{14}$$

This gives us the eccentricity ϵ in terms of l and ε:

$$\epsilon = \sqrt{1 - 4l^2\varepsilon} \tag{15}$$

and therefore in terms of L and E,

$$
\begin{aligned}
\epsilon &= \sqrt{1 + \frac{2EL^2}{\alpha^2\mu}} \\
p &= \frac{\mu L^2}{\alpha m}
\end{aligned}
$$

Collecting these results,

$$
\begin{array}{rl}
p = 2l^2 & \textit{semi latus rectum} \\
\epsilon = \sqrt{1 - 4\varepsilon l^2} & \textit{eccentricity} \\
a = \frac{p}{1-\epsilon^2} & \textit{semi major axis} \\
b = \frac{p}{\sqrt{1-\epsilon^2}} & \textit{semi minor axis}
\end{array}
$$

where ε and l are given by Eq.(9).

6.2 Unbound orbits

For unbound orbits in an inverse square force, there is no maximum separation of the particle from the center of force, so we take the impact parameter, $s = r_{min}$, as the reference point. The integration proceeds in the same way, but now we integrate out from r_{min},

$$\frac{\sqrt{2\alpha\mu}}{L}\varphi = -\int_{r_{min}}^{r} \frac{du}{\sqrt{u - \frac{L^2}{2\alpha\mu}u^2 + \frac{E}{\alpha}}}$$

The rescaled variables of Eq.(9) are now,

$$\varepsilon = \frac{E}{\alpha} > 0 \tag{16}$$

$$l^2 = \frac{L^2}{2\alpha\mu} \tag{17}$$

and the integration proceeds exactly as before, except that the angle φ_0 is defined as $\varphi_0 \equiv \sin^{-1}\left[\frac{1}{c}\left(\frac{l}{r_{min}} - \frac{1}{2l}\right)\right]$ onstead of $\varphi_0 \equiv \sin^{-1}\left(\frac{1}{c}\left(\frac{l}{r_{max}} - \frac{1}{2l}\right)\right)$. The solution is still

$$r = \frac{2l^2}{1 - 2lc\sin(\varphi - \varphi_0)}$$

The difference comes in setting the initial values. The minimum value of r occurs when $\varphi = 0$, so

$$r_{min} = \frac{2l^2}{1 + 2lc\sin\varphi_0}$$

and we find that $2lc\sin\varphi_0$ must take on a maximum value

$$2lc\sin\varphi_0 = |2lc|$$

Choosing c as the *positive* root

$$c = +\sqrt{\frac{1}{4l^2} + \varepsilon}$$

this requires $\sin\varphi_0 = 1$ and therefore, $\varphi_0 = \frac{\pi}{2}$. It follows from the definition of φ_0 that

$$\begin{aligned} \sin\frac{\pi}{2} = \sin\varphi_0 &= \frac{1}{c}\left(\frac{l}{r_{min}} - \frac{1}{2l}\right) \\ c &= \frac{l}{r_{min}} - \frac{1}{2l} \end{aligned}$$

and to have $c > 0$

$$2l^2 > r_{min} \tag{18}$$

Furthermore, with the orbit unbounded, we expect r to diverge at some value of $\varphi = \Psi$. This will happen provided

$$\begin{aligned} 1 - 2lc\sin(\Psi - \varphi_0) &= 0 \\ 1 - 2lc\cos\Psi &= 0 \\ \cos\Psi &= \frac{1}{2lc} \end{aligned}$$

and this requires a stronger inequality:

$$\begin{aligned} 1 &< 2lc \\ &= 2l\left(\frac{l}{s} - \frac{1}{2l}\right) \\ &= \frac{2l^2}{s} - 1 \end{aligned}$$

so that

$$\begin{aligned} \frac{2l^2}{r_{min}} - 1 > 1 \\ l^2 > r_{min} \end{aligned}$$

Now define

$$\epsilon \equiv \frac{2l^2}{r_{min}} - 1 > 1$$

so that

$$r = \frac{2l^2}{1 - \epsilon\cos\varphi} \tag{19}$$

Since we may write the ellipse in this same form by taking $\varphi = 0$ at r_{max} instead of r_{min}, the only difference between the bound and unbound orbits is whether $\varepsilon < 1$ for bound or $\varepsilon \geq 1$ for unbound. This observation allows us to examine all cases at once. The difference in sign shows up if we have a factor

$$\frac{l^2}{r_{min}} - 1$$

which is greater than zero for unbound orbits and less than zero for bound orbits.

6.3 Conic sections

6.3.1 Orbits

To see clearly the shapes of the possible orbits, we express the general solution in Cartesian coordinates. The coordinate transform is given by

$$\begin{aligned} x &= a + r\cos\varphi \\ y &= r\sin\varphi \\ r &= \sqrt{(x-a)^2 + y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x-a}\right) \end{aligned}$$

where a gives the offset between the focus and the center of the figure. Then Eq.(19) becomes

$$\begin{aligned} r - \epsilon r\cos\varphi &= 2l^2 \\ \sqrt{(x-a)^2 + y^2} - \epsilon(x-a) &= 2l^2 \\ \sqrt{(x-a)^2 + y^2} &= 2l^2 + \epsilon(x-a) \\ (x-a)^2 + y^2 &= 4l^4 + 4l^2\epsilon(x-a) + \epsilon^2(x-a)^2 \end{aligned}$$

We first note the one exceptional case when $1-\epsilon^2 = 0$. In this case the x^2 terms cancel, leaving

$$\begin{aligned} y^2 &= 4l^4 + 4l^2\epsilon(x-a) \\ x &= \frac{y^2}{4l^2\epsilon} - \frac{4l^4}{4l^2\epsilon} + a \end{aligned}$$

so that by choosing $a = \frac{l^2}{\epsilon}$ we have a parabola,

$$x = \frac{1}{4l^2\epsilon}y^2$$

This is the marginal unbound case between unbound and bound states, corresponding to the minimum energy required to escape to infinity.

Now return to the generic case with $\epsilon \neq 1$. Collect the $x-a$ terms and complete the square, noting carefully that $\left(1-\epsilon^2\right) = \pm\left|1-\epsilon^2\right|$ where the top sign gives a bound orbit and the bottom sign an unbound orbit.

$$\begin{aligned} \left(1-\epsilon^2\right)(x-a)^2 - 4l^2\epsilon(x-a) + y^2 &= 4l^4 \\ \pm\left(\left|1-\epsilon^2\right|(x-a)^2 \mp 4l^2\epsilon(x-a)\right) + y^2 &= 4l^4 \\ \pm\left(\sqrt{\left|1-\epsilon^2\right|}(x-a) \mp \frac{2l^2\epsilon}{\sqrt{\left|1-\epsilon^2\right|}}\right)^2 \mp \frac{4l^4\epsilon^2}{\left|1-\epsilon^2\right|^2} + y^2 &= 4l^4 \end{aligned}$$

Next choose the offset $a = \mp\frac{2l^2\epsilon}{\left|1-\epsilon^2\right|}$ so that

$$\begin{aligned} \pm\left|1-\epsilon^2\right|x^2 + y^2 &= 4l^4 \pm \frac{4l^4\epsilon^2}{\left|1-\epsilon^2\right|^2} \\ \pm\left|1-\epsilon^2\right|x^2 + y^2 &= 4l^4\frac{\left|1-\epsilon^2\right|^2 \pm \epsilon^2}{\left|1-\epsilon^2\right|^2} \end{aligned}$$

Now look at the two cases. Taking the top sign for the bound orbits, we define

$$\begin{aligned} \frac{1}{a^2} &\equiv \frac{\left|1-\epsilon^2\right|^3}{4l^4\left(\left|1-\epsilon^2\right|^2+\epsilon^2\right)} > 0 \\ \frac{1}{b^2} &\equiv \frac{\left|1-\epsilon^2\right|^2}{4l^4\left(\left|1-\epsilon^2\right|^2+\epsilon^2\right)} > 0 \end{aligned}$$

Then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

giving the equation of an ellipse in Cartesian coordinates. For the unbound case, recalling that $\epsilon > 1$,

$$\begin{aligned} -\left(\epsilon^2-1\right)x^2+y^2 &= 4l^4\frac{\left(\epsilon^2-1\right)^2-\epsilon^2}{\left(\epsilon^2-1\right)^2} \\ -\left(\epsilon^2-1\right)x^2+y^2 &= 4l^4\frac{\epsilon^4-3\epsilon^2+1}{\left(\epsilon^2-1\right)^2} \end{aligned}$$

The constant on the right has one zero with $\epsilon > 1$ at $\epsilon^2 = \frac{1}{2}\left(3+\sqrt{5}\right)$. This value makes the right side vanish so that

$$y = \pm x\sqrt{\epsilon^2-1}$$

These are asymptotes for hyperbolas. When the right side is greater than zero, defining constants as before, the equation takes the form

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

while for the right side negative, we have

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$$

Both solutions represent pairs of hyperbolas, with one pair opening right and left and the other opening up and down.

The ellipses, hyperbolas, parabolas and asymptotic solutions for the inverse square force law motion are precisely the figures that result from the possible intersections of a plane with a cone.

6.4 Alternative solutions using additional conserved quantitites

The brute force integration to find the equation of motion may be avoided if we begin with either of two alternative constants of the motion: Hamilton's vector, or the Laplace-Runge-Lenz vector.

6.4.1 New conserved quantities

From the angular momentum and the energy we may construct two other conserved quantities. Start with a pair of unit vectors in the xy plane

$$\begin{aligned} \hat{\mathbf{r}} &= \mathbf{i}\cos\varphi+\mathbf{j}\sin\varphi \\ \hat{\boldsymbol{\varphi}} &= -\mathbf{i}\sin\varphi+\mathbf{j}\cos\varphi \end{aligned}$$

The time rate of change of the unit vector $\hat{\boldsymbol{\varphi}}$ is given by

$$\begin{aligned}
\frac{d}{dt}\hat{\boldsymbol{\varphi}} &= \frac{d}{dt}\left(-\mathbf{i}\sin\varphi + \mathbf{j}\cos\varphi\right) \\
&= \left(-\mathbf{i}\cos\varphi - \mathbf{j}\sin\varphi\right)\dot{\varphi} \\
&= -\dot{\varphi}\hat{\mathbf{r}}
\end{aligned}$$

and therefore, using $L = \mu r^2 \dot{\varphi}$, we have

$$\begin{aligned}
\frac{d}{dt}\hat{\boldsymbol{\varphi}} &= -\frac{L}{\mu r^2}\hat{\mathbf{r}} \\
&= \frac{L}{\mu\alpha}\mathbf{F} \\
&= \frac{d}{dt}\left(\frac{L}{\mu\alpha}\mathbf{p}\right)
\end{aligned}$$

where the force is given by $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \equiv -\frac{\alpha}{r^2}\hat{\mathbf{r}}$. Then we have

$$\frac{d}{dt}\left(\mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}\right) = 0$$

Therefore, *Hamilton's vector*,

$$\mathbf{h} \equiv \mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}} \tag{20}$$

is conserved as a consequence of rotational invariance.

Since angular momentum is conserved, the product $\mathbf{A} \equiv \mathbf{h} \times \mathbf{L}$ is also conserved. Rearranging one of the resulting triple cross products,

$$\begin{aligned}
\mathbf{A} &\equiv \mathbf{h} \times \mathbf{L} \\
&= \left(\mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}\right) \times \mathbf{L} \\
&= \mathbf{p} \times \mathbf{L} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}} \times (\mathbf{r} \times \mathbf{p}) \\
&= \mathbf{p} \times \mathbf{L} - \frac{\mu\alpha}{L}\left(\mathbf{r}\left(\hat{\boldsymbol{\varphi}} \cdot \mathbf{p}\right) - \mathbf{p}\left(\hat{\boldsymbol{\varphi}} \cdot \mathbf{r}\right)\right) \\
&= \mathbf{p} \times \mathbf{L} - \frac{\mu\alpha}{L}\left(\mu r^2 \dot{\varphi}\right)\hat{\mathbf{r}}
\end{aligned}$$

to give

$$\mathbf{A} \equiv \mathbf{p} \times \mathbf{L} - \mu\alpha\hat{\mathbf{r}} \tag{21}$$

as a second conserved quantity. This is the *Laplace-Runge-Lenz vector.*

Either Hamilton's vector or the Laplace-Runge-Lenz vector can be used to easily get the solution for the motion.

6.4.2 Hamilton's vector

To find $r(\varphi)$ from Hamilton's vector, choose the initial conditions so that at time $t = 0$ the particle lies at perihelion, r_{min}, at $\varphi = 0$. Let this point lie on the x-axis. This is a turning point, so $\dot{r} = 0$ and the velocity is purely in the y-direction,

$$\mathbf{v} = v_0\mathbf{j} = r_{min}\dot{\varphi}_0\mathbf{j}$$

Then we may evaluate Hamilton's vector,

$$\begin{aligned}
\mathbf{h}(t) = \mathbf{h}(t=0) &= \mathbf{p}(0) - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}(0) \\
&= \left(\mu r_{min}\dot{\varphi}_0 - \frac{\mu\alpha}{L}\right)\hat{\mathbf{j}}
\end{aligned}$$

Since $\mathbf{h}(t) = \mathbf{h}(0)$, we may dot with $\hat{\boldsymbol{\varphi}}(t)$ at any later time t,

$$\begin{aligned}
\mathbf{h}(0)\cdot\hat{\boldsymbol{\varphi}}(t) &= \mathbf{h}(t)\cdot\hat{\boldsymbol{\varphi}}(t) \\
\left(\mu r_{min}\dot{\varphi}_0 - \frac{\mu\alpha}{L}\right)\mathbf{j}\cdot\hat{\boldsymbol{\varphi}} &= \left(\mathbf{p} - \frac{\mu\alpha}{L}\hat{\boldsymbol{\varphi}}\right)\cdot\hat{\boldsymbol{\varphi}} \\
\left(\mu r_{min}\dot{\varphi}_0 - \frac{\mu\alpha}{L}\right)\cos\varphi &= \mu r\dot{\varphi} - \frac{\mu\alpha}{L} \\
\left(\frac{L}{r_{min}} - \frac{\mu\alpha}{L}\right)\cos\varphi &= \frac{L}{r} - \frac{\mu\alpha}{L}
\end{aligned}$$

and therefore, solving for $\frac{1}{r}$,

$$\left(\frac{1}{r_{min}} - \frac{\mu\alpha}{L^2}\right)\cos\varphi + \frac{\mu\alpha}{L^2} = \frac{1}{r}$$

we invert to find

$$\begin{aligned}
r &= \frac{1}{\frac{\mu\alpha}{L^2} + \left(\frac{1}{r_{min}} - \frac{\mu\alpha}{L^2}\right)\cos\varphi} \\
&= \frac{L^2}{\mu\alpha}\frac{1}{1 + \left(\frac{L^2}{\mu\alpha r_{min}} - 1\right)\cos\varphi}
\end{aligned}$$

thereby recovering the equation for an ellipse.

6.4.3 Solving for the motion using the Laplace-Runge-Lenz vector

With $\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\hat{\mathbf{r}}$ constant, the equation of motion also follows quickly. Take the dot product with $\mathbf{r}$, and use the cyclic symmetry of the triple product,

$$\mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) = \mathbf{p}\cdot(\mathbf{L}\times\mathbf{r}) = \mathbf{L}\cdot(\mathbf{r}\times\mathbf{p})$$

to write

$$\begin{aligned}
\mathbf{r}\cdot\mathbf{A} &= \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - mk\mathbf{r}\cdot\hat{\mathbf{r}} \\
rA\cos\theta &= \mathbf{L}\cdot(\mathbf{r}\times\mathbf{p}) - mkr \\
&= L^2 - mkr \\
r(A\cos\theta + mk) &= L^2
\end{aligned}$$

and we again recover the equation for an ellipse.

$$r = \frac{L^2}{mk + A\cos\theta}$$

Putting this in standard form

$$r = \frac{p}{1 + \varepsilon\cos\varphi}$$

and identifying $\theta = \varphi$, we immediately have

$$
\begin{aligned}
p &= \frac{L^2}{mk} \\
\varepsilon &= \frac{A}{mk}
\end{aligned}
$$

and therefore

$$A = mk\sqrt{1 + \frac{2Ep}{\alpha}}$$

6.4.4 A note on the Laplace-Runge-Lenz vector

The Kepler problem begins with 6 coordinate degrees of freedom and 6 velocities, but reduces to an effective 1-dimensional problem with two initial conditions. This suggests that there are 10 symmetries and their corresponding conserved quantities. Six of these are given by the total momentum and total angular momentum, and one by the conserved energy. The remaining three may be assigned to the Laplace-Runge-Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$

We check its constancy directly,

$$
\begin{aligned}
\frac{d\mathbf{A}}{dt} &= \dot{\mathbf{p}} \times \mathbf{L} - mk\frac{d}{dt}(\hat{\mathbf{r}}) \\
&= \dot{\mathbf{p}} \times (\mathbf{r} \times \mathbf{p}) - mk\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \\
&= -\frac{k}{r^2}\hat{\mathbf{r}} \times (\mathbf{r} \times \mathbf{p}) - mk\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \mathbf{r}\frac{\dot{r}}{r^2}\right) \\
&= -\frac{k}{r^2}(\mathbf{r}p_r - r\mathbf{p}) - mk\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \mathbf{r}\frac{\dot{r}}{r^2}\right) \\
&= -\frac{k}{r^2}\left(\mathbf{r}m\dot{r} - rm\frac{d\mathbf{r}}{dt}\right) - mk\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \mathbf{r}\frac{\dot{r}}{r^2}\right) \\
&= -km\mathbf{r}\frac{\dot{r}}{r^2} + \frac{mk}{r}\frac{d\mathbf{r}}{dt} - \frac{mk}{r}\frac{d\mathbf{r}}{dt} + mk\mathbf{r}\frac{\dot{r}}{r^2} \\
&= 0
\end{aligned}
$$

Suppose only assume that the force in the Laplace-Runge-Lenz vector is central, replacing $\dot{\mathbf{p}} = \mathbf{F} = -F\hat{\mathbf{r}}$ without specifying F further. Then

$$
\begin{aligned}
\frac{d\mathbf{A}}{dt} &= \dot{\mathbf{p}} \times (\mathbf{r} \times \mathbf{p}) - mk\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \\
&= \mathbf{F} \times (\mathbf{r} \times \mathbf{p}) - mk\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \mathbf{r}\frac{\dot{r}}{r^2}\right) \\
&= -F\left(\mathbf{r}m\dot{r} - rm\frac{d\mathbf{r}}{dt}\right) - mk\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \mathbf{r}\frac{\dot{r}}{r^2}\right) \\
&= -Fm\dot{r}\mathbf{r} + mk\mathbf{r}\frac{\dot{r}}{r^2} + Fmr\frac{d\mathbf{r}}{dt} - mk\frac{1}{r}\frac{d\mathbf{r}}{dt} \\
&= -\left(F - \frac{k}{r^2}\right)m\dot{r}\mathbf{r} + \left(F - \frac{k}{r^2}\right)mr\frac{d\mathbf{r}}{dt} \\
&= m\left(F - \frac{k}{r^2}\right)\left(r\frac{d\mathbf{r}}{dt} - \mathbf{r}\dot{r}\right)
\end{aligned}
$$

Now look at the last factor,

$$
\begin{aligned}
r\frac{d\mathbf{r}}{dt}-\mathbf{r}\dot{r} &= r\left(\frac{d\mathbf{r}}{dt}-\hat{\mathbf{r}}\dot{r}\right) \\
&= r\left(\hat{\mathbf{r}}\dot{r}+\hat{\boldsymbol{\varphi}}r\dot{\varphi}-\hat{\mathbf{r}}\dot{r}\right) \\
&= r^2\dot{\varphi}\hat{\boldsymbol{\varphi}} \\
&= \frac{L}{m}\hat{\boldsymbol{\varphi}} \\
&= \frac{1}{m}\mathbf{L}\times\hat{\mathbf{r}}
\end{aligned}
$$

so we may write

$$
\frac{d\mathbf{A}}{dt} = \left(F-\frac{k}{r^2}\right)\mathbf{L}\times\hat{\mathbf{r}}
$$

This shows that the conservation of the vector is specific to an inverse square force law.

7 Perturbation theory

Perturbations of known solutions give useful information while only requiring us to solve linear equations. Often perturbation can give us highly accurate answers without having the full exact expression. As an example, we begin with a simple constraint problem, then address the effects of non-Newtonian potentials.

7.1 Motion constrained to a paraboloid

Consider a small, frictionless bead constrained to move on the inside surface of a paraboloid, given in cylindrical coordinates by

$$z=\alpha\rho^2$$

We may describe the motion of the bead by starting with the unconstrained action for the motion of the bead in a gravitational field,

$$S_0=\int\left(\frac{1}{2}m\left(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2\right)-mgz\right)$$

and adding the constraint. To simplify the expressions, we write the constraint as $m\left(z-\alpha\rho^2\right)=0$. With Lagrange multiplier λ, the action becomes

$$S=\int\left(\frac{1}{2}m\left(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2\right)-mgz+m\lambda\left(z-\alpha\rho^2\right)\right)dt$$

Varying with respect to ρ,φ,z and λ, and cancelling common factors of m, the equations of motion are

$$
\begin{aligned}
0 &= \ddot{\rho}-\rho\dot{\varphi}^2+2\lambda\alpha\rho \\
0 &= \ddot{z}+g-\lambda \\
L &= m\rho^2\dot{\varphi} \\
0 &= z-\alpha\rho^2
\end{aligned}
\tag{22}
$$

7.1.1 Zeroth order solution

The perturbation theory begins by expanding the coordinates and functions of the coordinates in a series of the form

$$q=q_0+\varepsilon q_1+\varepsilon^2 q_2+\cdots$$

where the q_i are functions of time and $\varepsilon \ll 1$. For the paraboloid problem to first order, let

$$\begin{aligned}\rho &= \rho_0 + \varepsilon\rho_1 \\ z &= z_0 + \varepsilon z_1 \\ \dot{\varphi} &= \omega_0 + \varepsilon\omega_1 \\ \lambda &= \lambda_0 + \varepsilon\lambda_1\end{aligned} \tag{23}$$

For the zeroth order we seek a circular motion, so that ρ_0, z_0 and ω_0 are constant. Substituting into the equations of motions, Eqs.(22), the time derivatives all vanish, leaving

$$\begin{aligned}0 &= -\rho_0\omega_0^2 + 2\lambda\alpha\rho_0 \\ 0 &= g - \lambda_0 \\ L &= m\rho_0^2\omega_0 \\ 0 &= z_0 - \alpha\rho_0^2\end{aligned}$$

and therefore for any ρ_0,

$$\begin{aligned}\lambda_0 &= g \\ \omega_0 &= \sqrt{2\alpha g} \\ z_0 &= \alpha\rho_0^2 \\ L_0 &= m\rho_0^2\sqrt{2\alpha g}\end{aligned} \tag{24}$$

and we have a solution for any value of ρ_0. It is convenient (though not necessary) to expand $L = L_0 + \varepsilon L_1$ as well.

7.1.2 First order solution

At first order, we substitute the full first order expansion, Eqs.(23), into the equations of motion. With the zeroth order terms given by Eqs.(24), and keeping only zeroth and first order terms,

$$\begin{aligned}0 &= \varepsilon\ddot{\rho}_1 - (\rho_0 + \varepsilon\rho_1)(\omega_0 + \varepsilon\omega_1)^2 + 2(g + \varepsilon\lambda_1)\alpha(\rho_0 + \varepsilon\rho_1) \\ &= \varepsilon\ddot{\rho}_1 - \rho_0\omega_0^2 - 2\varepsilon\rho_0\omega_0\omega_1 - \varepsilon\rho_1\omega_0^2 + 2g\alpha\rho_0 + 2\varepsilon\lambda_1\alpha\rho_0 + 2g\alpha\varepsilon\rho_1 \\ &= \rho_0\left(2g\alpha - \omega_0^2\right) + \varepsilon\left(\ddot{\rho}_1 - 2\rho_0\omega_0\omega_1 - \rho_1\omega_0^2 + 2\lambda_1\alpha\rho_0 + 2g\alpha\rho_1\right)\end{aligned}$$

$$0 = \varepsilon\ddot{z}_1 + g - g - \varepsilon\lambda_1$$

$$L = m\rho_0^2\omega_0 + m\varepsilon\left(\rho_0^2\omega_1 + 2\rho_0\omega_0\rho_1\right)$$

$$0 = \left(z_0 - \alpha\rho_0^2\right) + \varepsilon\left(z_1 - 2\alpha\rho_0\rho_1\right)$$

Imposing the zeroth order solution leaves expressions linear in ε, so for the first order perturbation we must solve

$$\begin{aligned}0 &= \ddot{\rho}_1 - 2\rho_0\omega_0\omega_1 - \rho_1\omega_0^2 + 2\lambda_1\alpha\rho_0 + 2g\alpha\rho_1 \\ 0 &= \ddot{z}_1 - \lambda_1 \\ 0 &= \rho_0^2\omega_1 + 2\rho_0\omega_0\rho_1 \\ 0 &= z_1 - 2\alpha\rho_0\rho_1\end{aligned}$$

Differentiating the constraint

$$z_1 = 2\alpha\rho_0\rho_1$$

twice we have $\ddot{z}_1 = 2\alpha\rho_0\ddot{\rho}_1$ and therefore,

$$\lambda_1 = 2\alpha\rho_0\ddot{\rho}_1$$

while the third equation gives

$$\omega_1 = -\frac{2\omega_0}{\rho_0}\rho_1$$

This reduces everything to a single linear equation for ρ_1,

$$\begin{aligned}
0 &= \ddot{\rho}_1 - 2\rho_0\omega_0\omega_1 - \rho_1\omega_0^2 + 2\lambda_1\alpha\rho_0 + 2g\alpha\rho_1 \\
&= \ddot{\rho}_1 - 2\rho_0\omega_0\left(-\frac{2\omega_0}{\rho_0}\rho_1\right) - \rho_1\omega_0^2 + 2\left(2\alpha\rho_0\ddot{\rho}_1\right)\alpha\rho_0 + 2g\alpha\rho_1 \\
&= \left(1 + 4\alpha^2\rho_0^2\right)\ddot{\rho}_1 + \left(4\omega_0^2 - \omega_0^2 + 2g\alpha\right)\rho_1 \\
&= \left(1 + 4\alpha^2\rho_0^2\right)\ddot{\rho}_1 + 4\omega_0^2\rho_1
\end{aligned}$$

Choosing $\rho_1 = 0$ at $t = 0$, we have the immediate solution,

$$\rho_1 = A\sin\omega t$$

where

$$\omega = \frac{2\omega_0}{\sqrt{1 + 4\alpha^2\rho_0^2}}$$

This gives the full solution for the first order corrections. Putting the zeroth and first order terms together, we have at this order,

$$\begin{aligned}
\rho &= \rho_0 + \varepsilon A\sin\omega t \\
z &= \alpha\rho_0^2 + 2\varepsilon\alpha\rho_0 A\sin\omega t \\
&\approx \alpha\left(\rho_0 + \varepsilon A\sin\omega t\right)^2 \\
\dot{\varphi} &= \omega_0\left(1 - \frac{2A\varepsilon}{\rho_0}\sin\omega t\right) \\
\varphi &= \omega_0\left(t + \frac{2A\varepsilon}{\rho_0\omega}\cos\omega t\right) \\
\lambda &= g - 2\varepsilon\alpha\rho_0 A\omega^2\sin\omega t \\
L &= m\rho_0^2\omega_0 = L_0
\end{aligned}$$

where

$$\omega \equiv \frac{2\omega_0}{\sqrt{1 + 2\alpha\rho_0}}$$

The energy, keeping only up to first order terms, is given by

$$\begin{aligned}
E &= \frac{1}{2}m\left(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{z}^2\right)+mgz-m\lambda\left(z-\alpha\rho^2\right) \\
&= \frac{1}{2}m\left(\left(\varepsilon\omega A\sin\omega t\right)^2+\left(\rho_0+\varepsilon A\sin\omega t\right)^2\left(\omega_0\left(1-\frac{2A\varepsilon}{\rho_0}\sin\omega t\right)\right)^2+\left(2\varepsilon\alpha\rho_0\omega A\cos\omega t\right)^2\right) \\
&\quad +mg\left(\alpha\rho_0^2+2\varepsilon\alpha\rho_0 A\sin\omega t\right)-m\left(g-2\varepsilon\alpha\rho_0 A\omega^2\sin\omega t\right)\left(\left(\alpha\rho_0^2+2\varepsilon\alpha\rho_0 A\sin\omega t\right)-\alpha\left(\rho_0+\varepsilon A\sin\omega t\right)^2\right) \\
&= \frac{1}{2}m\left(\rho_0^2+2\varepsilon\rho_0 A\sin\omega t\right)\omega_0^2\left(1-\frac{4A\varepsilon}{\rho_0}\sin\omega t\right) \\
&\quad +mg\alpha\rho_0^2+2\varepsilon mg\alpha\rho_0 A\sin\omega t-m\left(g-2\varepsilon\alpha\rho_0 A\omega^2\sin\omega t\right)\left(\alpha\rho_0^2+2\varepsilon\alpha\rho_0 A\sin\omega t-\alpha\rho_0^2-2\alpha\varepsilon\rho_0 A\sin\omega t\right) \\
&= \frac{1}{2}m\omega_0^2\left(\rho_0^2+2\varepsilon\rho_0 A\sin\omega t-4\rho_0 A\varepsilon\sin\omega t\right)+mg\alpha\rho_0^2+2\varepsilon mg\alpha\rho_0 A\sin\omega t \\
&= \left(\frac{1}{2}m\rho_0^2\omega_0^2+mg\alpha\rho_0^2\right)+\varepsilon Am\rho_0\left(2g\alpha-\omega_0^2\right)\sin\omega t
\end{aligned}$$

and since $\omega_0^2=2g\alpha$,

$$E = E_0$$

Notice that even if the energy were to change, it must still be a constant. The oscillatory term had to cancel. It should seem odd that the energy and angular momentum stay the same, because their values establish the uniqueness of the solution – given E, L there is only one motion possible! The resolution of this seeming paradox is that these constants of the motion *do* change, but only at quadratic order in perturbation.

7.2 Perihelion advance in non-Newtonian potentials

Let the Newtonian gravitational potential be modified to

$$V(r)=-\frac{\alpha}{r}f(r)$$

and suppose we can expand f in inverse powers of r,

$$f(r)=\sum_{n=0}^{\infty}\frac{a_n\varepsilon^n}{r^n}$$

with $a_0=1$ and $\frac{\varepsilon}{r}$ small. We set $a_1=a$ and $a_2=b$ and keep only the first two new terms,

$$f\approx 1+\varepsilon\frac{a}{r}+\varepsilon^2\frac{b}{r^2}$$

It is a useful let ε have units of *length* while a and b are dimensionless. We assume $\frac{\varepsilon}{r}\ll 1$ and a, b are of order unity.

The action is

$$\begin{aligned}
S &= \int_0^t\left(\frac{1}{2}\mu\left(\dot{r}^2+r^2\dot{\varphi}^2\right)-V(r)\right)dt \\
&= \int_0^t\left(\frac{1}{2}\mu\left(\dot{r}^2+r^2\dot{\varphi}^2\right)+\frac{\alpha}{r}+\varepsilon\frac{\alpha a}{r^2}+\varepsilon^2\frac{\alpha b}{r^3}\right)dt
\end{aligned}$$

where $\mu=\frac{mM}{M+m}$ is the reduced mass and $L=\mu r^2\dot{\varphi}$ the conserved angular momentum.

The equations of motion are

$$\begin{aligned} -\mu\ddot{r}+\mu r\dot{\varphi}^2-\frac{\alpha}{r^2}-\frac{2\alpha a\varepsilon}{r^3}-\frac{3\alpha b\varepsilon^2}{r^4} &= 0 \\ \frac{d}{dt}\left(\mu r^2\dot{\varphi}\right) &= 0 \end{aligned}$$

and we have conserved angular momentum and energy

$$\begin{aligned} L &= \mu r^2\dot{\varphi} \\ E &= \frac{1}{2}\mu\left(\dot{r}^2+r^2\dot{\varphi}^2\right)-\frac{\alpha}{r}-\frac{\alpha a\varepsilon}{r^2}-\frac{\alpha b\varepsilon^2}{r^3} \\ &= \frac{1}{2}\mu\dot{r}^2+\frac{L^2}{2\mu r^2}-\frac{\alpha}{r}-\frac{\alpha a\varepsilon}{r^2}-\frac{\alpha b\varepsilon^2}{r^3} \end{aligned}$$

We expand the coordinates as

$$\begin{aligned} r &= r_0+\varepsilon r_1+\varepsilon_2 r_2+\cdots \\ \dot{\varphi} &= \omega_0+\varepsilon\omega_1+\varepsilon_2\omega_2+\cdots \end{aligned}$$

where we use $\dot{\varphi}$ since φ is cyclic.

As with the paraboloid, we begin our perturbation we start with the $\varepsilon = 0$ solution. We already know that this will be an ellipse, but here we will restrict to a circular orbit as the zeroth order solution. We easily see that circular orbits satisfy the equations of motion. These will have constant $r = r_0$ and $\dot{\varphi} = \omega_0$. These values must be related by

$$L = \mu r_0^2\omega_0 \tag{25}$$

$$E = \frac{L^2}{2\mu r_0^2}-\frac{\alpha}{r_0} \tag{26}$$

To satisfy the equations of motion, we must also have

$$\begin{aligned} 0 &= -\mu\ddot{r}_0+\mu r_0\dot{\varphi}^2-\frac{\alpha}{r_0^2} \\ &= \mu r_0\omega_0^2-\frac{\alpha}{r_0^2} \end{aligned}$$

so solving for the frequency,

$$\omega_0^2 = \frac{\alpha}{\mu r_0^3}$$

We recognize this as Kepler's law.

We choose initial conditions such that at $t = 0$, $\dot{r} = 0$, $r = r_{min}$ and $\varphi = 0$. Then with the energy and angular momentum given by Eqs.(26) and (25) we find

$$L = \mu r_0^2\omega_0 = \mu r^2\dot{\varphi}$$

the initial angular velocity $\dot{\varphi}$ is ω_0.

7.3 First order perturbation

At order ε, the radius and angular velocity will vary. Keeping up to first order terms,

$$\begin{aligned} r &= r_0+\varepsilon r_1 \\ \dot{\varphi} &= \omega_0+\varepsilon\omega_1 \end{aligned}$$

Substituting these into the first order equations of motion

$$\begin{aligned}
0 &= -\mu\ddot{r}+\mu r\dot{\varphi}^2-\frac{\alpha}{r^2}-\frac{2\alpha a\varepsilon}{r^3} \\
&= -\varepsilon\mu\ddot{r_1}+\mu\left(r_0+\varepsilon r_1\right)\left(\omega_0+\varepsilon\omega_1\right)^2-\frac{\alpha}{\left(r_0+\varepsilon r_1\right)^2}-\frac{2\alpha a\varepsilon}{\left(r_0+\varepsilon r_1\right)^3} \\
&= -\varepsilon\mu\ddot{r_1}+\mu\left(r_0\omega_0^2+2\varepsilon r_0\omega_0\omega_1+\varepsilon r_1\omega_0^2\right)-\frac{\alpha}{r_0^2}\left(1+\varepsilon\frac{r_1}{r_0}\right)^{-2}-\frac{2\alpha a\varepsilon}{r_0^3}\left(1+\varepsilon\frac{r_1}{r_0}\right)^{-3} \\
&= \left(\mu r_0\omega_0^2-\frac{\alpha}{r_0^2}\right)-\varepsilon\left(\mu\ddot{r_1}-2\mu r_0\omega_0\omega_1-\mu\omega_0^2 r_1-\frac{2\alpha}{r_0^3}r_1+\frac{2\alpha a}{r_0^3}\right)
\end{aligned}$$

$$\begin{aligned}
L &= \mu r^2\dot{\varphi} \\
&= \mu\left(r_0+\varepsilon r_1\right)^2\left(\omega_0+\varepsilon\omega_1\right) \\
&= \mu\left(r_0^2+2\varepsilon r_0 r_1\right)\left(\omega_0+\varepsilon\omega_1\right) \\
&= \mu r_0^2\omega_0+\varepsilon\mu r_0^2\omega_1+2\varepsilon\mu r_0\omega_0 r_1
\end{aligned}$$

With $L=\mu r_0^2\omega_0$ the angular momentum equation gives

$$\omega_1 = -\frac{2r_1}{r_0}\omega_0$$

As expected, the zeroth order terms of the radial equation are identically satisfied. Substituting ω_1 into the remaining first order part,

$$\begin{aligned}
0 &= \mu\ddot{r_1}-2\mu r_0\omega_0\left(-\frac{2r_1}{r_0}\omega_0\right)-\mu\omega_0^2 r_1-\frac{2\alpha}{r_0^3}r_1+\frac{2\alpha a}{r_0^3} \\
&= \mu\ddot{r_1}+\left(4\mu\omega_0^2-\mu\omega_0^2-\frac{2\alpha}{r_0^3}\right)r_1+\frac{2\alpha a}{r_0^3}
\end{aligned}$$

Here we see an overall shift in the average radius of the orbit. This occurs because we required the angular momentum to stay the same. Let

$$r_1 = y + A$$

with the constant A given by

$$\begin{aligned}
\left(4\mu\omega_0^2-\mu\omega_0^2-\frac{2\alpha}{r_0^3}\right)A &= -\frac{2\alpha a}{r_0^3} \\
A &= -\frac{2\alpha a}{3\mu\omega_0^2 r_0^3-2\alpha}
\end{aligned}$$

With $\alpha=\mu r_0^3\omega_0^2$ the awkward fraction becomes

$$\begin{aligned}
\frac{2\alpha a}{3\mu\omega_0^2 r_0^3-2\alpha} &= \frac{2\mu r_0^3\omega_0^2 a}{3\mu\omega_0^2 r_0^3-2\mu r_0^3\omega_0^2} \\
&= 2a
\end{aligned}$$

and we have $A=-2a$.

Then the equation becomes simple harmonic,

$$\mu\ddot{y}+\left(4\mu\omega_0^2-\mu\omega_0^2-\frac{2\alpha}{r_0^3}\right)y = 0$$

so that

$$y = B_1\sin\omega t+B_2\cos\omega t$$

with squared frequency,

$$\begin{aligned}\omega^2 &= 3\omega_0^2 - \frac{2\alpha}{\mu r_0^3} \\ &= 3\omega_0^2 - 2\omega_0^2 \\ &= \omega_0^2\end{aligned}$$

The full solution to first order is therefore,

$$\begin{aligned}r &= (r_0 - 2a\varepsilon) + \varepsilon\left(B_1 \sin\omega_0 t + B_2 \cos\omega_0 t\right) \\ &= \tilde{r}_0 + \varepsilon\left(B_1 \sin\omega_0 t + B_2 \cos\omega_0 t\right) \\ \dot{\varphi} &= \omega_0\left(1 - \varepsilon\left(\frac{2}{r_0}\left(B_1 \sin\omega_0 t + B_2 \cos\omega_0 t\right) - 4a\right)\right) \\ &= \omega_0\left(1 + 2\left(r_0 - \tilde{r}_0\right) - \frac{2\varepsilon}{r_0}\left(B_1 \sin\omega_0 t + B_2 \cos\omega_0 t\right)\right)\end{aligned}$$

Now check the initial conditions. We have required $r = r_{min}$ at $t = 0$, so

$$r_{min} = \tilde{r}_0 + \varepsilon B_2$$

At this point, the radial velocity vanishes so that

$$\dot{r} = 0 = \varepsilon B_1$$

and since B_2 must be negative to give r_{min} we set $B_2 = -R$. Then

$$\begin{aligned}r &= \tilde{r}_0 - \varepsilon R \cos\omega_0 t \\ \dot{\varphi} &= \omega_0\left(1 + 2\left(r_0 - \tilde{r}_0\right) + \frac{2\varepsilon}{r_0} R \cos\omega_0 t\right)\end{aligned}$$

making the angular velocity maximum at r_{min}.

Finally, we integrate $\dot{\varphi}$ to find $\varphi(t)$,

$$\begin{aligned}\frac{d\varphi}{dt} &= \omega_0\left(1 + 2\left(r_0 - \tilde{r}_0\right) + \frac{2\varepsilon}{r_0} R \cos\omega_0 t\right) \\ \int_0^{\varphi} d\varphi &= \int_0^{t} \omega_0\left(1 + 2\left(1 - \frac{\tilde{r}_0}{r_0}\right) + \frac{2\varepsilon}{r_0} R \cos\omega_0 t\right) dt \\ \varphi &= \left(1 + 2\left(1 - \frac{\tilde{r}_0}{r_0}\right)\right)\omega_0 t + \frac{2\varepsilon}{r_0} R \sin\omega_0 t\end{aligned}$$

where

$$\begin{aligned}1 - \frac{\tilde{r}_0}{r_0} &= 1 - \frac{1}{r_0}\left(r_0 - \varepsilon\frac{2\alpha a}{3\mu\omega_0^2 r_0^3 - 2\alpha}\right) \\ &= \frac{2a\varepsilon}{r_0}\end{aligned}$$

Therefore,

$$\varphi = \left(1 + \frac{4a\varepsilon}{r_0}\right)\omega_0 t + \frac{2\varepsilon}{r_0} R \sin\omega_0 t$$

As required, this vanishes at $t = 0$.

7.3.1 Perihelion advance

The period τ is the time it takes the orbiting particle to pass through a full circuit, $\varphi = 2\pi$, so

$$2\pi = \left(1 + \frac{4a\varepsilon}{r_0}\right)\omega_0\tau + \frac{2\varepsilon}{r_0}R\sin\omega_0\tau$$

Since τ will be close to $\frac{2\pi}{\omega_0}$ we set

$$\tau = \frac{2\pi}{\omega_0} + \Delta\tau$$

and expand with $\Delta\tau$ small,

$$\begin{aligned} 2\pi &= \left(1 + \frac{4a\varepsilon}{r_0}\right)\omega_0\left(\frac{2\pi}{\omega_0} + \Delta\tau\right) + \frac{2\varepsilon}{r_0}R\sin\omega_0\left(\frac{2\pi}{\omega_0} + \Delta\tau\right) \\ 2\pi &\approx 2\pi + \omega_0\Delta\tau + \frac{8\pi a\varepsilon}{r_0} \end{aligned}$$

and therefore,

$$\Delta\tau = -\frac{8\pi a\varepsilon}{r_0\omega_0}$$

However, r now oscillates as

$$r = \tilde{r}_0 - \varepsilon R\cos\omega_0 t$$

so the perihelion is achieved in a time

$$t_p = \frac{2\pi}{\omega_0}$$

The ratio of the time to perihelion to the orbital period is therefore,

$$\begin{aligned} \frac{t_p}{\tau} &= \frac{\frac{2\pi}{\omega_0}}{\frac{2\pi}{\omega_0} + \Delta\tau} \\ &= \frac{1}{1 - \frac{4a\varepsilon}{r_0}} \\ &\approx 1 + \frac{4a\varepsilon}{r_0} \end{aligned}$$

and perihelion occurs later by $\frac{4a\varepsilon\tau}{r_0}$. The angular advance of perihelion is then

$$\begin{aligned} \Delta\varphi &= \omega\Delta\tau \\ &= \omega_0\Delta\tau \\ &= -\frac{8\pi a\varepsilon}{r_0} \end{aligned}$$

where $V = -\frac{\alpha}{r} - \frac{\varepsilon\alpha a}{r^2}$.

8 Example: Schwarzschild geodesics

We can use these classical considerations to get an approximate estimate of perihelion advance in general relativity.

General relativity gives a slight correction to the inverse square law. The motion is governed by the geodesic equation, which the radial motion of an orbit at $\theta = \frac{\pi}{2}$ is

$$\frac{d^2r}{d\tau^2} = -\left(1 - \frac{a}{r}\right)\frac{ac^2}{2r^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{a}{2r^2\left(1 - \frac{a}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{a}{r}\right)r\left(\frac{d\varphi}{d\tau}\right)^2 \quad (27)$$

where $a = \frac{2GM}{c^2}$, and $u^\alpha = \frac{dx^\alpha}{d\tau}$, $\alpha = 0, 1, 2, 3$. Proper time, τ is related to t by

$$\frac{du^0}{d\tau} = -\frac{a}{r^2\left(1-\frac{a}{r}\right)} u^0 u^1$$

where $u^0 = c\frac{dt}{d\tau}$ and $u^1 = \frac{dr}{d\tau}$. Integrating to find $\frac{dt}{d\tau}$,

$$\begin{aligned}
\frac{du^0}{d\tau} &= -\frac{a}{r^2\left(1-\frac{a}{r}\right)} u^0 u^1 \\
\frac{1}{u^0}\frac{du^0}{d\tau} &= -\frac{a}{r\,(r-a)}\frac{dr}{d\tau} \\
\int \frac{du^0}{u^0} &= -\int \frac{a\,dr}{r\,(r-a)} \\
\ln\left(\frac{u^0}{b}\right) &= \int dr\left(\frac{1}{r} - \frac{1}{r-a}\right) \\
&= \ln\left(\frac{r}{r-a}\right) \\
c\frac{dt}{d\tau} = u^0 &= \frac{b}{1-\frac{a}{r}}
\end{aligned}$$

at $r \to \infty$, $t = \tau$ so $b = 1$, and we have

$$\frac{dt}{d\tau} = \frac{1}{1-\frac{a}{r}}$$

For the velocities, $u^\alpha = \frac{dx^\alpha}{d\tau}$, we simply have

$$\frac{dx^\alpha}{d\tau} = \frac{dt}{d\tau}\frac{dx^\alpha}{dt} = \frac{1}{1-\frac{a}{r}}\frac{dx^\alpha}{dt}$$

For the acceleration $\frac{d^2r}{d\tau^2}$,

$$\begin{aligned}
\frac{d^2r}{d\tau^2} &= \frac{d}{d\tau}\frac{dr}{d\tau} \\
&= \frac{dt}{d\tau}\frac{d}{dt}\left(\frac{dt}{d\tau}\frac{dr}{dt}\right) \\
&= \frac{1}{1-\frac{a}{r}}\frac{d}{dt}\left(\frac{1}{1-\frac{a}{r}}\frac{dr}{dt}\right) \\
&= \frac{1}{1-\frac{a}{r}}\left(\frac{1}{1-\frac{a}{r}}\frac{d^2r}{dt^2} + \frac{1}{\left(1-\frac{a}{r}\right)^2}\frac{a}{r^2}\frac{dr}{dt}\frac{dr}{dt}\right) \\
&= \frac{1}{\left(1-\frac{a}{r}\right)^2}\frac{d^2r}{dt^2} + \frac{1}{\left(1-\frac{a}{r}\right)^3}\frac{a}{r^2}\frac{dr}{dt}\frac{dr}{dt}
\end{aligned}$$

Sutstituting into Eq.(27),

$$\begin{aligned}
\frac{1}{\left(1-\frac{a}{r}\right)^2}\frac{d^2r}{dt^2} + \frac{1}{\left(1-\frac{a}{r}\right)^3}\frac{a}{r^2}\frac{dr}{dt}\frac{dr}{dt} &= \frac{1}{\left(1-\frac{a}{r}\right)^2}\left(-\left(1-\frac{a}{r}\right)\frac{ac^2}{2r^2}\left(\frac{dt}{dt}\right)^2 + \frac{a}{2r^2\left(1-\frac{a}{r}\right)}\left(\frac{dr}{dt}\right)^2 + \left(1-\frac{a}{r}\right)r\left(\frac{d\varphi}{dt}\right)^2\right) \\
\frac{d^2r}{dt^2} &= -\left(1-\frac{a}{r}\right)\frac{ac^2}{2r^2} - \frac{1}{1-\frac{a}{r}}\frac{a}{r^2}\left(\frac{dr}{dt}\right)^2 + \frac{a}{2r^2\left(1-\frac{a}{r}\right)}\left(\frac{dr}{dt}\right)^2 + \left(1-\frac{a}{r}\right)r\left(\frac{d\varphi}{dt}\right)^2
\end{aligned}$$

Now consider the force on a particle initially at rest. Then the final three terms on the right vanish and we have

$$F_{grav} = m\frac{d^2r}{dt^2} = -\left(1-\frac{a}{r}\right)\frac{amc^2}{2r^2}$$

With $a = \frac{2GM}{c^2}$, the lowest order term reproduces Newtonian gravity,

$$F_{grav}|_{lowest\ order} = -\frac{GMm}{r^2}$$

but there is a small correction,

$$F_{grav} = -\frac{GMm}{r^2}\left(1-\frac{2GM}{rc^2}\right)$$

where the additional term is the classical escape velocity over the speed of light,

$$\frac{2GM}{rc^2} = \frac{v_{excape}^2}{c^2} \ll 1$$

The potential for the relativistic force is

$$V_{GR} = -\frac{GMm}{r} + \frac{G^2M^2m}{r^2c^2}$$

Comparing this to the general form of potential derived above,

$$\begin{aligned}
-\frac{\alpha}{r} - \frac{\varepsilon\alpha a}{r^2} &= -\frac{GMm}{r} + \frac{G^2M^2m}{r^2c^2} \\
\alpha &= GMm \\
\varepsilon a &= -\frac{GM}{c^2}
\end{aligned}$$

Then using our expression for the perihelion advance,

$$\begin{aligned}
\Delta\varphi &= -\frac{8\pi a\varepsilon}{r_0} \\
&= \frac{8\pi GM}{r_0c^2}
\end{aligned}$$

so the perihelion advances. The actual expression, calculated entirely within general relativity, is about $\Delta\varphi = \frac{6\pi GM}{r_0c^2}$.

9 Non-power-law force: the Yukawa potential

We consider properties of the Yukawa potential,

$$V(r) = \frac{k}{r}e^{-\frac{r}{a}}$$

This potential is the static, spherically symmetric solution to the Klein-Gordon equation,

$$-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + \nabla^2 V = \frac{m^2c^2}{\hbar^2}V$$

To see this, let $V = V(r)$ and write the Laplacian in spherical coordinates. Then for $V = V(r)$ the angular derivatives drop out and we have $\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)$. Differentiating on the left side, the Klein-Gordon equation becomes

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} - \frac{m^2c^2}{\hbar^2}V = 0 \tag{28}$$

Let $U = rV$. Then

$$\begin{aligned}\frac{d^2U}{dr^2} &= \frac{d}{dr}\left(V + r\frac{dV}{dr}\right) \\ &= 2\frac{dV}{dr} + r\frac{d^2V}{dr^2}\end{aligned}$$

so in terms of U, Eq.(28) becomes

$$\begin{aligned}\frac{1}{r}\frac{d^2U}{dr^2} - \frac{m^2c^2}{\hbar^2}\frac{U}{r} &= 0 \\ \frac{d^2U}{dr^2} - \frac{m^2c^2}{\hbar^2}U &= 0\end{aligned}$$

and this has exponential solutions,

$$U = U_+ \exp\left(+\frac{mc}{\hbar}r\right) + U_- \exp\left(-\frac{mc}{\hbar}r\right)$$

Choosing the attractive potential for our solution, we have

$$V = -\frac{k}{r}e^{-\frac{mc}{\hbar}r}$$

where $a = \frac{\hbar}{mc}$ is the reduced Compton wavelength of a particle of mass m.

Because the range of the potential decreases exponentially, the resulting force is very short range, and negligibly small if $r \gg \frac{\hbar}{mc}$. Furthermore, unless $r \gg \frac{\hbar}{mc}$, where $\frac{\hbar}{mc}$ is the Compton wavelength, a quantum treatment is necessary. Therefore, the Klein-Gordon field is of little relevance classically. Here we study it merely as an example of a non-power-law potential.

The action for a particle moving in this potential is

$$S = \int \left(\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{k}{r}e^{-\frac{r}{a}}\right)$$

so the equations of motion are

$$\begin{aligned}-\mu\ddot{r} + \mu r\dot{\varphi}^2 - \frac{k}{r^2}\left(1 + \frac{r}{a}\right)e^{-\frac{r}{a}} &= 0 \\ \mu r^2\dot{\varphi} &= L\end{aligned}$$

9.1 Bound orbits

We know from our general results that the conserved energy and angular momentum are given by

$$\begin{aligned}E &= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r}e^{-\frac{r}{a}} \\ L &= \mu r^2\dot{\varphi}\end{aligned}$$

and from the energy expression we see that the radial motion is described by the effective potential

$$V_{eff} = \frac{L^2}{2\mu r^2} - \frac{k}{r}e^{-\frac{r}{a}}$$

Bound orbits exist if there is a minimum of the effective potential. The extrema are given by

$$\begin{aligned}0 &= \frac{dV_{eff}}{dr} \\ &= -\frac{L^2}{\mu r^3} + \frac{k}{r^2}e^{-\frac{r}{a}} + \frac{k}{ar}e^{-\frac{r}{a}} \\ &= -\frac{L^2}{\mu r^3} + \left(\frac{k}{r^2} + \frac{k}{ar}\right)e^{-\frac{r}{a}}\end{aligned}$$

This has solutions if and only if

$$\frac{L^2}{\mu} = kr_0\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}} \tag{29}$$

The right side vanishes at $r_0 = 0$, vanishes as $r_0 \to \infty$, and is positive definite, so there is always a value of L small enough that the equation is satisfied. To find the number of extrema with r_0 positive, we look at first and second derivatives, of the right hand side, $f(r) = kr\left(1+\frac{r}{a}\right)e^{-\frac{r}{a}}$:

$$\begin{aligned}\frac{df}{dr} &= \left(k+\frac{2kr}{a}-\frac{kr}{a}-\frac{kr^2}{a^2}\right)e^{-\frac{r}{a}} \\ &= k\left(1+\frac{r}{a}-\frac{r^2}{a^2}\right)e^{-\frac{r}{a}}\end{aligned}$$

This vanishes if and only if

$$\begin{aligned}-r^2+ar+a^2 &= 0 \\ r &= -\frac{1}{2}\left(-a\pm\sqrt{a^2+4a^2}\right)\end{aligned}$$

so the only positive solution is $r_{max} = \frac{a}{2}\left(1+\sqrt{5}\right) > 0$. The second derivative is

$$\begin{aligned}\frac{d^2f}{dr^2} &= \frac{k}{a}\left(1-\frac{2r}{a}-1-\frac{r}{a}+\frac{r^2}{a^2}\right)e^{-\frac{r}{a}} \\ &= \frac{kr}{a^2}\left(-3+\frac{r}{a}\right)e^{-\frac{r}{a}}\end{aligned}$$

so there is an inflexion point at $r = 3a$ as $f(r)$ enters into the exponential tail. Otherwise, the function is concave downward and there is a single maximum.

We conclude that if

$$\frac{L^2}{\mu} = kr_{max}\left(1+\frac{r_{max}}{a}\right)e^{-\frac{r_{max}}{a}}$$

then there is exactly one bound state at r_{max}, while for

$$\frac{L^2}{\mu} < kr_{max}\left(1+\frac{r_{max}}{a}\right) = e^{-\frac{r_{max}}{a}}ka\left(2+\sqrt{5}\right)e^{-\frac{1}{2}\left(1+\sqrt{5}\right)}$$

there will be two solutions for positive r.

9.2 Circular orbits

Now consider the precession of nearly circular orbits in the Yukawa potential,

$$V(r) = -\frac{k}{r}e^{-\frac{r}{a}}$$

We know from our general results that the conserved energy and angular momentum are given by

$$\begin{aligned}E &= \frac{1}{2}\mu\dot{r}^2+\frac{L^2}{2\mu r^2}-\frac{k}{r}e^{-\frac{r}{a}} \\ L &= \mu r^2\dot{\varphi}\end{aligned}$$

For circular orbits, $r = r_0$, given by Eq.(29), so

$$\begin{aligned} E_0 &= \frac{L^2}{2\mu r_0^2} - \frac{k}{r_0} e^{-\frac{r_0}{a}} \\ L_0 &= \mu r_0^2 \dot{\varphi}_0 \end{aligned}$$

Eq.(29) gives L in terms of r_0, and using this to eliminate the dependence on L, the energy becomes

$$\begin{aligned} E_0 &= \frac{1}{2r_0^2} \left(k r_0 \left(1 + \frac{r_0}{a} \right) e^{-\frac{r_0}{a}} \right) - \frac{k}{r_0} e^{-\frac{r_0}{a}} \\ &= -\frac{k}{2r_0} \left(1 - \frac{r_0}{a} \right) e^{-\frac{r_0}{a}} \end{aligned}$$

Therefore, both L and E are determined by r_0,

$$\begin{aligned} E_0 &= -\frac{k}{2r_0} \left(1 - \frac{r_0}{a} \right) e^{-\frac{r_0}{a}} \\ L_0^2 &= \mu k r_0 \left(1 + \frac{r_0}{a} \right) e^{-\frac{r_0}{a}} \end{aligned}$$

This also gives us the frequency of the circular orbit,

$$\begin{aligned} \omega_0^2 &= \dot{\varphi}_0^2 \\ &= \frac{L^2}{\mu^2 r_0^4} \\ &= \frac{k}{\mu r_0^3} \left(1 + \frac{r_0}{a} \right) e^{-\frac{r_0}{a}} \end{aligned}$$

9.3 Perturbed orbit

Now suppose we give the system slightly higher energy by instantaneously increasing the angular momentum by a small fraction, ε, so that $L = L_0 (1 + \varepsilon)$. Then expanding the coordinates of the particle about the circular orbit,

$$\begin{aligned} r &= r_0 + \varepsilon \xi \\ \dot{\varphi} = \omega^2 &= \frac{L_0^2 (1 + 2\varepsilon)}{\mu^2 (r_0 + \varepsilon \xi)^4} \\ &= \frac{L_0^2}{\mu^2 r_0^4} (1 + 2\varepsilon) \left(1 + \frac{\varepsilon \xi}{r_0} \right)^{-4} \\ &= \omega_0^2 \left(1 + 2\varepsilon - \frac{4\varepsilon \xi}{r_0} \right) \end{aligned}$$

This solves the angular equation for ω, leaving only the radial equation, $-\frac{k}{2r_0}\left(1-\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}=E_0$

$$\begin{aligned}
0 &= -\mu\ddot{r}+\mu r\dot{\varphi}^2-\frac{k}{r^2}\left(1+\frac{r}{a}\right)e^{-\frac{r}{a}} \\
&= -\mu\ddot{r}+\frac{L^2}{\mu r^3}-\frac{k}{r^2}\left(1+\frac{r}{a}\right)e^{-\frac{r}{a}} \\
&= -\varepsilon\mu\ddot{\xi}+\frac{L_0^2\left(1+2\varepsilon\right)}{\mu\left(r_0+\varepsilon\xi\right)^3}-\left(\frac{k}{\left(r_0+\varepsilon\xi\right)^2}+\frac{k}{a\left(r_0+\varepsilon\xi\right)}\right)e^{-\frac{r_0+\varepsilon\xi}{a}} \\
&= -\varepsilon\mu\ddot{\xi}+\frac{L_0^2}{\mu r_0^3}\left(1+2\varepsilon\right)\left(1-\frac{3\varepsilon\xi}{r_0}\right)-\left(\frac{k}{r_0^2}\left(1-\frac{2\varepsilon\xi}{r_0}\right)+\frac{k}{ar_0}\left(1-\frac{\varepsilon\xi}{r_0}\right)\right)e^{-\frac{r_0}{a}}e^{-\frac{\varepsilon\xi}{a}} \\
&= -\varepsilon\mu\ddot{\xi}+\frac{L_0^2}{\mu r_0^3}\left(1+2\varepsilon-\frac{3\varepsilon\xi}{r_0}\right)-\left(\frac{k}{r_0^2}+\frac{k}{ar_0}-\frac{2k\varepsilon\xi}{r_0^3}-\frac{\varepsilon k\xi}{ar_0^2}\right)e^{-\frac{r_0}{a}}\left(1-\frac{\varepsilon\xi}{a}\right) \\
&= \frac{L_0^2}{\mu r_0^3}-\left(\frac{k}{r_0^2}+\frac{k}{ar_0}\right)e^{-\frac{r_0}{a}}-\varepsilon\mu\ddot{\xi}+\varepsilon\frac{L_0^2}{\mu r_0^3}\left(2-\frac{3\xi}{r_0}\right)+\frac{\varepsilon k\xi}{r_0^3}\left(2+\frac{2r_0}{a}+\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}} \\
&= E_0-\varepsilon\left(\mu\ddot{\xi}-\mu r_0\omega_0^2\left(2-\frac{3\xi}{r_0}\right)-\frac{k\xi}{r_0^3}\left(2+\frac{2r_0}{a}+\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}\right)
\end{aligned}$$

so we must solve

$$\ddot{\xi}+\left(3\omega_0^2-\frac{k}{\mu r_0^3}\left(2+\frac{2r_0}{a}+\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}\right)\xi = 2r_0\omega_0^2$$

Setting $\chi=\xi+A$ where

$$\begin{aligned}
A &= \frac{2r_0\omega_0^2}{3\omega_0^2-\frac{k}{\mu r_0^3}\left(2+\frac{2r_0}{a}+\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}} \\
&= \frac{2r_0\frac{k}{\mu r_0^3}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}}{3\frac{k}{\mu r_0^3}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}-\frac{k}{\mu r_0^3}\left(2+\frac{2r_0}{a}+\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}} \\
&= \frac{2r_0\left(1+\frac{r_0}{a}\right)}{1+\frac{r_0}{a}-\frac{r_0^2}{a^2}}
\end{aligned}$$

removes the constant term on the right, leaving

$$\begin{aligned}
\ddot{\chi}+\left(3\omega_0^2-\frac{k}{\mu r_0^3}\left(2+\frac{2r_0}{a}+\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}\right)\chi &= 0 \\
\ddot{\chi}+\frac{k}{\mu r_0^3}\left(1+\frac{r_0}{a}-\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}\chi &= 0
\end{aligned}$$

and we see that the radius oscillates in simple harmonic motion with frequency

$$\omega_r=\sqrt{\frac{k}{\mu r_0^3}\left(1+\frac{r_0}{a}-\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}}$$

where the orbital frequency is

$$\omega^2 = \omega_0^2\left(1+2\varepsilon-\frac{4\varepsilon\xi}{r_0}\right) \approx \omega_0^2 = \frac{k}{\mu r_0^3}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}$$

so comparing the radial and orbital frequencies,

$$\begin{aligned}
\frac{\omega_r^2}{\omega_0^2} &= \frac{\frac{k}{\mu r_0^3}\left(1+\frac{r_0}{a}-\frac{r_0^2}{a^2}\right)e^{-\frac{r_0}{a}}}{\frac{k}{\mu r_0^3}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}} \\
&= \frac{1+\frac{r_0}{a}-\frac{r_0^2}{a^2}}{1+\frac{r_0}{a}} \\
&= \frac{a^2+ar_0-r_0^2}{a\left(a+r_0\right)}
\end{aligned}$$

Since the numerator must be nonnegative, we require

$$\begin{aligned}
a^2+ar_0-r_0^2 &\geq 0 \\
r_0 &\leq \frac{1+\sqrt{5}}{2}a
\end{aligned}$$

Finally, the energy changes to

$$\begin{aligned}
E &= \frac{1}{2}\mu\varepsilon\dot{\xi}^2+\frac{\left(L_0+\varepsilon\right)^2}{2\mu\left(r_0+\varepsilon\xi\right)^2}-\frac{k}{\left(r_0+\varepsilon\xi\right)}e^{-\frac{r_0+\varepsilon\xi}{a}} \\
&= \frac{1}{2}\mu\varepsilon\dot{\xi}^2+\frac{L_0^2\left(1+\frac{2\varepsilon}{L_0}\right)}{2\mu r_0^2}\left(1-\frac{2\varepsilon\xi}{r_0}\right)-\frac{k}{r_0}\left(1-\frac{\varepsilon\xi}{r_0}\right)e^{-\frac{r_0}{a}\left(1+\frac{\varepsilon\xi}{r_0}\right)} \\
&= \frac{1}{2}\mu\varepsilon\dot{\xi}^2+\frac{L_0^2}{2\mu r_0^2}\left(1+\frac{2\varepsilon}{L_0}-\frac{2\varepsilon\xi}{r_0}\right)-\frac{k}{r_0}\left(1-\frac{\varepsilon\xi}{r_0}\right)e^{-\frac{r_0}{a}\left(1+\frac{\varepsilon\xi}{r_0}\right)} \\
&= \frac{k}{2r_0}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}+\varepsilon\left(\frac{1}{2}\mu\dot{\xi}^2+\frac{k}{2r_0}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}\left(\frac{2}{L_0}-\frac{2\xi}{r_0}\right)\right)-\frac{k}{r_0}\left(1-\frac{\varepsilon\xi}{r_0}\right)e^{-\frac{r_0}{a}}e^{-\frac{\varepsilon\xi}{a}} \\
&= \frac{k}{2r_0}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}+\varepsilon\left(\frac{1}{2}\mu\dot{\xi}^2+\frac{k}{2r_0}\left(1+\frac{r_0}{a}\right)e^{-\frac{r_0}{a}}\left(\frac{2}{L_0}-\frac{2\xi}{r_0}\right)\right)-\frac{k}{r_0}e^{-\frac{r_0}{a}}\left(1-\frac{\varepsilon\xi}{r_0}-\frac{\varepsilon\xi}{a}\right) \\
&= E_0+\varepsilon\left(\frac{1}{2}\mu\dot{\xi}^2+\frac{k}{r_0L_0}e^{-\frac{r_0}{a}}\left(1+\frac{r_0}{a}\right)\right)
\end{aligned}$$

10 Bertrand's Theorem

10.1 Circular orbits

The effective potential,

$$V_{eff}=\frac{L_\varphi^2}{2\mu r^2}+V\left(r\right)$$

has a minimum or maximum at r_0 if and only if

$$\begin{aligned}
0 &= \left.\frac{dV_{eff}}{dr}\right|_{r_0} \\
&= -\frac{L_\varphi^2}{\mu r_0^3}+\left.\frac{dV}{dr}\right|_{r_0} \\
&= -\frac{L_\varphi^2}{\mu r_0^3}-f\left(r_0\right)
\end{aligned}$$

so we must have

$$f\left(r_0\right)=-\frac{L_\varphi^2}{\mu r_0^3}$$

At this radius, there is no net radial force, so that circular orbits are possible. Such orbits may be stable or unstable, depending on the sign of the second derivative of the effective potential. Stable orbits occur only if

$$\begin{aligned} 0 &< \left.\frac{d^2 V_{eff}}{dr^2}\right|_{r_0} \\ &= \frac{3L_\varphi^2}{\mu r_0^4} + \left.\frac{d^2 V}{dr^2}\right|_{r_0} \\ &= \frac{3L_\varphi^2}{\mu r_0^4} - \frac{df}{dr}(r_0) \end{aligned}$$

and since $f(r_0) = -\frac{L_\varphi^2}{\mu r_0^3}$, we have

$$\frac{df}{dr}(r_0) < -\frac{3}{r_0} f(r_0)$$

10.2 General case

Eliminating $\dot{\varphi} = \frac{L_\varphi}{\mu r^2}$, the remaining, radial equation of motion is

$$\mu \ddot{r} - \frac{L_\varphi^2}{\mu r^3} - f(r) = 0$$

Write this as an orbit equation, using

$$\begin{aligned} \dot{r} &= \frac{d\varphi}{dt}\frac{dr}{d\varphi} \\ &= \frac{L_\varphi}{\mu r^2}\frac{dr}{d\varphi} \\ \ddot{r} &= \frac{d\varphi}{dt}\frac{d}{d\varphi}\left(\frac{L_\varphi}{\mu r^2}\frac{dr}{d\varphi}\right) \\ &= \frac{L_\varphi}{\mu r^2}\left(-\frac{2L_\varphi}{\mu r^3}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi}{\mu r^2}\frac{d^2 r}{d\varphi^2}\right) \\ &= -\frac{2L_\varphi^2}{\mu^2 r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi^2}{\mu^2 r^4}\frac{d^2 r}{d\varphi^2} \end{aligned}$$

Substituting,

$$\begin{aligned} 0 &= \mu \ddot{r} - \frac{L_\varphi^2}{\mu r^3} - f(r) \\ &= -\frac{2L_\varphi^2}{\mu r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi^2}{\mu r^4}\frac{d^2 r}{d\varphi^2} - \frac{L_\varphi^2}{\mu r^3} - f(r) \end{aligned}$$

Now let $r = \frac{1}{u}$

$$\begin{aligned} \frac{dr}{d\varphi} &= -\frac{1}{u^2}\frac{du}{d\varphi} \\ \frac{d^2 r}{d\varphi^2} &= \frac{d}{d\varphi}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right) \\ &= \frac{2}{u^3}\left(\frac{du}{d\varphi}\right)^2 - \frac{1}{u^2}\frac{d^2 u}{d\varphi^2} \end{aligned}$$

The equation of motion now becomes

$$\begin{aligned}
0 &= -\frac{2L_\varphi^2}{\mu r^5}\left(\frac{dr}{d\varphi}\right)^2+\frac{L_\varphi^2}{\mu r^4}\frac{d^2r}{d\varphi^2}-\frac{L_\varphi^2}{\mu r^3}-f(r) \\
&= -\frac{2L_\varphi^2u^5}{\mu}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right)^2+\frac{L_\varphi^2u^4}{\mu}\left(\frac{2}{u^3}\left(\frac{du}{d\varphi}\right)^2-\frac{1}{u^2}\frac{d^2u}{d\varphi^2}\right)-\frac{L_\varphi^2}{\mu r^3}-f(r) \\
&= -\frac{2L_\varphi^2u}{\mu}\left(\frac{du}{d\varphi}\right)^2+\frac{2L_\varphi^2u}{\mu}\left(\frac{du}{d\varphi}\right)^2-\frac{L_\varphi^2u^2}{\mu}\frac{d^2u}{d\varphi^2}-\frac{L_\varphi^2u^3}{\mu}-f(r) \\
&= -\frac{L_\varphi^2u^2}{\mu}\frac{d^2u}{d\varphi^2}-\frac{L_\varphi^2u^3}{\mu}-f(r)
\end{aligned}$$

or

$$\frac{d^2u}{d\varphi^2}+u=-\frac{\mu}{L_\varphi^2u^2}f\left(\frac{1}{u}\right)$$

Finally, write the force in terms of the potential,

$$\begin{aligned}
f\left(\frac{1}{u}\right) &= -\frac{dV}{dr} \\
&= -\frac{du}{dr}\frac{dV}{du} \\
&= u^2\frac{dV}{du}
\end{aligned}$$

Now we have simply

$$\frac{d^2u}{d\varphi^2}+u=-\frac{\mu}{L_\varphi^2}\frac{dV}{du}$$

This is the equation of a driven harmonic oscillator. This striking set of transformations is what happens when people spend 300 years working on a problem.

Now consider circular orbits. This means that u does not change with φ at all, so we have $\frac{d^2u}{d\varphi^2}=0$ and therefore, if we set

$$h(u)=-\frac{\mu}{L_\varphi^2}\frac{dV}{du}$$

then

$$u_0=h(u_0)$$

Next, expand u and the force for small perturbations about u_0,

$$\begin{aligned}
u &= u_0+\eta \\
-\frac{\mu}{L_\varphi^2}\frac{dV}{du} &= h(u) \\
&= h(u_0)+h'(u_0)\eta+\frac{1}{2}h''(u_0)\eta^2+\cdots
\end{aligned}$$

Substituting these into the equation of motion,

$$\begin{aligned}
\frac{d^2\eta}{d\varphi^2}+u_0+\eta &= h(u_0)+h'(u_0)\eta+\frac{1}{2}h''(u_0)\eta^2+\cdots \\
&= u_0+h'(u_0)\eta+\frac{1}{2}h''(u_0)\eta^2+\cdots \\
\frac{d^2\eta}{d\varphi^2}+\left[1-h'(u_0)\right]\eta &= \frac{1}{2}h''(u_0)\eta^2+\cdots
\end{aligned}$$

If $1-h'(u_0)<0$, then the equation has exponential instead of oscillatory solutions, and the circular orbits are not stable. For stable orbits, we therefore set

$$\lambda^2 = 1 - h'(u_0) > 0$$

If we neglect the quadratic and higher terms on the right side of the equation, with initial condition $\eta = 0$ when $\varphi = 0$, we have solutions

$$\eta = A \sin \lambda\varphi$$

In order to reproduce the initial conditions after some integer number, q, of complete orbits, we require

$$\begin{aligned} \eta(0) &= \eta(2\pi q) \\ 0 &= A \sin 2\pi q\lambda \end{aligned}$$

so that

$$q\lambda = p$$

with p another integer. We see that λ must be rational,

$$\lambda = \frac{p}{q}$$

Assuming the force is a continuous function of position, h' is continuous and λ is also continuous. Therefore, $\lambda = \frac{p}{q}$ for *all* nearly circular orbits. Returning to the definition of λ, and using the constancy of λ,

$$\begin{aligned} \lambda^2 &= 1 - h'(u_0) \\ &= 1 - \frac{d}{du}\left(-\frac{\mu}{L_\varphi^2 u^2} f\left(\frac{1}{u}\right)\right) \\ \lambda^2 - 1 &= -\frac{2\mu}{L_\varphi^2 u_0^3} f\left(\frac{1}{u_0}\right) + \frac{\mu}{L_\varphi^2 u^2} f'\left(\frac{1}{u_0}\right) \end{aligned}$$

Since the circular orbit satisfies

$$\begin{aligned} u_0 &= h(u_0) \\ &= -\frac{\mu}{L_\varphi^2 u_0^2} f\left(\frac{1}{u_0}\right) \\ f\left(\frac{1}{u_0}\right) &= -\frac{L_\varphi^2}{\mu} u_0^3 \end{aligned}$$

so that

$$\begin{aligned} \lambda^2 - 1 &= -\frac{2\mu}{L_\varphi^2 u_0^3}\left(-\frac{L_\varphi^2}{\mu} u_0^3\right) + \frac{\mu}{L_\varphi^2 u^2} f'\left(\frac{1}{u_0}\right) \\ \lambda^2 - 1 &= 2 + \frac{\mu}{L_\varphi^2 u_0^2}\left(\frac{\left(-\frac{L_\varphi^2}{\mu} u_0^3\right)}{f\left(\frac{1}{u_0}\right)}\right) f'\left(\frac{1}{u_0}\right) \\ \lambda^2 - 3 &= -\frac{u_0}{f\left(\frac{1}{u_0}\right)} f'\left(\frac{1}{u_0}\right) \end{aligned}$$

This must hold regardless of the value of u_0, so we may drop the subscript and integrate

$$\begin{aligned}
-\frac{df}{f} &= \left(\lambda^2 - 3\right)\frac{du}{u} \\
-\frac{df}{f} &= \left(\lambda^2 - 3\right)\left(-\frac{1}{r^2}\right) r dr \\
\frac{df}{f} &= \left(\lambda^2 - 3\right)\frac{dr}{r} \\
f(r) &= A r^{\lambda^2 - 3}
\end{aligned}$$

and in order to have stable, perturbatively closed orbits, the force law must be a rational power law. This gives h as

$$\begin{aligned}
h &= \frac{\mu}{L_\varphi^2 u^2}\frac{A}{u^{\lambda^2-3}} \\
&= \frac{\mu A}{L_\varphi^2} u^{1-\lambda^2}
\end{aligned}$$

Now return to the full equation of motion

$$\frac{d^2\eta}{d\varphi^2} + u_0 + \eta = u_0 + h'(u_0)\,\eta + \frac{1}{2}h''(u_0)\,\eta^2 + \frac{1}{6}h'''(u_0)\,\eta^3 + \cdots$$

and expand in a Fourier series,

$$\eta(\varphi) = \eta_0 + \eta_1 \cos\lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots$$

We also need powers of η. Keeping up to third order,

$$\begin{aligned}
\left(\eta_0 + \eta_1 \cos\lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots\right)^2 &= \eta_0^2 + \eta_0\eta_1 \cos\lambda\varphi + \eta_0\eta_2 \cos 2\lambda\varphi + \eta_0\eta_3 \cos 3\lambda\varphi \\
&\quad + \eta_0\eta_1 \cos\lambda\varphi + \eta_1^2 \cos^2\lambda\varphi + \eta_2\eta_1 \cos\lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0\eta_2 \cos 2\lambda\varphi + \eta_1\eta_2 \cos 2\lambda\varphi \cos\lambda\varphi + \eta_0\eta_3 \cos 3\lambda\varphi \\
\left(\eta_0 + \eta_1 \cos\lambda\varphi + \eta_2 \cos 2\lambda\varphi + \eta_3 \cos 3\lambda\varphi + \cdots\right)^3 &= \eta_0^3\eta_0 + \eta_0^2\eta_1 \cos\lambda\varphi + \eta_0^2\eta_2 \cos 2\lambda\varphi + \eta_0^2\eta_3 \cos 3\lambda\varphi + \cdots \\
&\quad + \eta_0^2\eta_1 \cos\lambda\varphi + \eta_0\eta_1^2 \cos^2\lambda\varphi + \eta_2\eta_0\eta_1 \cos\lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0\eta_0^2\eta_2 \cos 2\lambda\varphi + \eta_1\eta_0\eta_2 \cos 2\lambda\varphi \cos\lambda\varphi \\
&\quad + \eta_0^2\eta_3 \cos 3\lambda\varphi \\
&\quad + \eta_0^2\eta_1 \cos\lambda\varphi + \eta_0\eta_1^2 \cos^2\lambda\varphi + \eta_0\eta_1\eta_2 \cos\lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0\eta_1^2 \cos^2\lambda\varphi + \eta_1^3 \cos^3\lambda\varphi \\
&\quad + \eta_0\eta_2\eta_1 \cos\lambda\varphi \cos 2\lambda\varphi \\
&\quad + \eta_0^2\eta_2 \cos 2\lambda\varphi + \eta_0\eta_1\eta_2 \cos 2\lambda\varphi \cos\lambda\varphi \\
&\quad + \eta_0\eta_1\eta_2 \cos 2\lambda\varphi \cos\lambda\varphi \\
&\quad + \eta_0^2\eta_3 \cos 3\lambda\varphi
\end{aligned}$$

and using addition formulas

$$\begin{aligned}
\cos^2\lambda\varphi &= \frac{1}{2}\left(1+\cos 2\lambda\varphi\right)\\
\cos\lambda\varphi\cos 2\lambda\varphi &= \frac{1}{2}\left[\cos\left(\lambda\varphi+2\lambda\varphi\right)+\cos\left(\lambda\varphi-2\lambda\varphi\right)\right]\\
&= \frac{1}{2}\left[\cos\left(3\lambda\varphi\right)+\cos\left(\lambda\varphi\right)\right]\\
\cos^3\lambda\varphi &= \cos\lambda\varphi\frac{1}{2}\left(1+\cos 2\lambda\varphi\right)\\
&= \frac{1}{2}\cos\lambda\varphi+\frac{1}{4}\cos\left(3\lambda\varphi\right)+\frac{1}{4}\cos\left(\lambda\varphi\right)\\
&= \frac{3}{4}\cos\lambda\varphi+\frac{1}{4}\cos\left(3\lambda\varphi\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right)^2 &= \eta_0^2+\frac{1}{2}\eta_1^2+\left(2\eta_0\eta_1+\eta_1\eta_2\right)\cos\left(\lambda\varphi\right)\\
&\quad+\left(2\eta_0\eta_2+\frac{1}{2}\eta_1^2\right)\cos 2\lambda\varphi+\left(2\eta_0\eta_3+\eta_1\eta_2\right)\cos 3\lambda\varphi\\
\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right)^3 &= \eta_0^3+\left(3\eta_0^2\eta_1\right)\cos\lambda\varphi+\left(3\eta_0^2\eta_2\right)\cos 2\lambda\varphi+\left(3\eta_0^2\eta_3\right)\cos 3\lambda\varphi\\
&\quad+3\eta_0\eta_1^2\cos^2\lambda\varphi+6\eta_0\eta_1\eta_2\cos 2\lambda\varphi\cos\lambda\varphi+\eta_1^3\cos^3\lambda\varphi\\
&= \eta_0^3+\left(3\eta_0^2\eta_1\right)\cos\lambda\varphi+\left(3\eta_0^2\eta_2\right)\cos 2\lambda\varphi+\left(3\eta_0^2\eta_3\right)\cos 3\lambda\varphi\\
&\quad+3\eta_0\eta_1^2\frac{1}{2}\left(1+\cos 2\lambda\varphi\right)+6\eta_0\eta_1\eta_2\frac{1}{2}\left[\cos\left(3\lambda\varphi\right)+\cos\left(\lambda\varphi\right)\right]\\
&\quad+\eta_1^3\frac{3}{4}\cos\lambda\varphi+\frac{1}{4}\eta_1^3\cos\left(3\lambda\varphi\right)\\
&= \eta_0^3+\frac{3}{2}\eta_0\eta_1^2+\left(3\eta_0^2\eta_1+3\eta_0\eta_1\eta_2+\frac{3}{4}\eta_1^3\right)\cos\lambda\varphi\\
&\quad+\left(3\eta_0^2\eta_2+\frac{3}{2}\eta_0\eta_1^2\right)\cos 2\lambda\varphi+\left(3\eta_0^2\eta_3+\frac{1}{4}\eta_1^3+3\eta_0\eta_1\eta_2\right)\cos 3\lambda\varphi
\end{aligned}$$

Substituting and expanding,

$$\begin{aligned}
\frac{d^2\eta}{d\varphi^2}+u_0+\eta &= u_0+h'\left(u_0\right)\eta+\frac{1}{2}h''\left(u_0\right)\eta^2+\frac{1}{6}h'''\left(u_0\right)\eta^3+\cdots\\
\left(-\eta_1\lambda^2\cos\lambda\varphi-4\lambda^2\eta_2\cos 2\lambda\varphi-9\lambda^2\eta_3\cos 3\lambda\varphi+\cdots\right)+\eta &= h'\left(u_0\right)\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right)\\
&\quad+\frac{1}{2}h''\left(u_0\right)\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right)^2\\
&\quad+\frac{1}{6}h'''\left(u_0\right)\left(\eta_0+\eta_1\cos\lambda\varphi+\eta_2\cos 2\lambda\varphi+\eta_3\cos 3\lambda\varphi+\cdots\right)^3
\end{aligned}$$

Equating terms of equal frequency,

$$\begin{aligned}
\eta_0 &= h'\left(u_0\right)\eta_0+\frac{1}{2}h''\left(u_0\right)\eta_0^2+\frac{1}{2}h''\left(u_0\right)\frac{1}{2}\eta_1^2+\frac{1}{6}h'''\left(u_0\right)\eta_0^3+\frac{1}{6}h'''\left(u_0\right)\frac{3}{2}\eta_0\eta_1^2\\
\left(-\eta_1\lambda^2+\eta_1\right)\cos\lambda\varphi &= \left[h'\left(u_0\right)\eta_1+\frac{1}{2}h''\left(u_0\right)\left(2\eta_0\eta_1+\eta_1\eta_2\right)+\frac{1}{6}h'''\left(u_0\right)\left(3\eta_0^2\eta_1+3\eta_0\eta_1\eta_2+\frac{3}{4}\eta_1^3\right)\right]\cos\lambda\varphi\\
\left(-4\lambda^2\eta_2+\eta_2\right)\cos 2\lambda\varphi &= \left[h'\left(u_0\right)\eta_2+\frac{1}{2}h''\left(u_0\right)\left(2\eta_0\eta_2+\frac{1}{2}\eta_1^2\right)+\frac{1}{6}h'''\left(u_0\right)\left(3\eta_0^2\eta_2+\frac{3}{2}\eta_0\eta_1^2\right)\right]\cos 2\lambda\varphi\\
\left(-9\lambda^2\eta_3+\eta_3\right)\cos 3\lambda\varphi &= \left[h'\left(u_0\right)\eta_3+\frac{1}{2}h''\left(u_0\right)\left(2\eta_0\eta_3+\eta_1\eta_2\right)+\frac{1}{6}h'''\left(u_0\right)\left(3\eta_0^2\eta_3+\frac{1}{4}\eta_1^3+3\eta_0\eta_1\eta_2\right)\right]\cos 3\lambda\varphi
\end{aligned}$$

We set $h'(u_0) = 1 - \lambda^2$ and cancel or combine the resulting like terms,

$$\begin{aligned}
0 &= -\lambda^2\eta_0 + \frac{1}{2}h''(u_0)\,\eta_0^2 + \frac{1}{2}h''(u_0)\,\frac{1}{2}\eta_1^2 + \frac{1}{6}h'''(u_0)\,\eta_0^3 + \frac{1}{6}h'''(u_0)\,\frac{3}{2}\eta_0\eta_1^2 \\
0 &= \frac{1}{2}h''(u_0)\,(2\eta_0\eta_1 + \eta_1\eta_2) + \frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_1 + 3\eta_0\eta_1\eta_2 + \frac{3}{4}\eta_1^3\right) \\
-3\lambda^2\eta_2 &= \frac{1}{2}h''(u_0)\left(2\eta_0\eta_2 + \frac{1}{2}\eta_1^2\right) + \frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_2 + \frac{3}{2}\eta_0\eta_1^2\right) \\
-8\lambda^2\eta_3 &= \frac{1}{2}h''(u_0)\,(2\eta_0\eta_3 + \eta_1\eta_2) + \frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_3 + \frac{1}{4}\eta_1^3 + 3\eta_0\eta_1\eta_2\right)
\end{aligned}$$

Now look at the first equation,

$$\begin{aligned}
0 &= -\lambda^2\eta_0 + \frac{1}{2}h''(u_0)\,\eta_0^2 + \frac{1}{2}h''(u_0)\,\frac{1}{2}\eta_1^2 + \frac{1}{6}h'''(u_0)\,\eta_0^3 + \frac{1}{6}h'''(u_0)\,\frac{3}{2}\eta_0\eta_1^2 \\
0 &= \eta_0\left(-\lambda^2 + \frac{1}{2}h''(u_0)\,\eta_0 + \frac{1}{6}h'''(u_0)\,\eta_0^2\right) + \left(\frac{1}{2}h''(u_0)\,\frac{1}{2} + \frac{1}{6}h'''(u_0)\,\frac{3}{2}\eta_0\right)\eta_1^2
\end{aligned}$$

We seek relationships between the different η_i, assuming all are small. Thus, to lowest order,

$$\begin{aligned}
\left(-\lambda^2 + \frac{1}{2}h''(u_0)\,\eta_0 + \frac{1}{6}h'''(u_0)\,\eta_0^2\right) &\approx -\lambda^2 \\
\left(\frac{1}{2}h''(u_0)\,\frac{1}{2} + \frac{1}{6}h'''(u_0)\,\frac{3}{2}\eta_0\right) &\approx \frac{1}{4}h''(u_0)
\end{aligned}$$

giving

$$\begin{aligned}
0 &= -\lambda^2\eta_0 + \frac{1}{4}h''(u_0)\,\eta_1^2 \\
\eta_0 &= \frac{1}{4\lambda^2}h''(u_0)\,\eta_1^2
\end{aligned}$$

For the second equation,

$$\begin{aligned}
0 &= \frac{1}{2}h''(u_0)\,(2\eta_0\eta_1 + \eta_1\eta_2) + \frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_1 + 3\eta_0\eta_1\eta_2 + \frac{3}{4}\eta_1^3\right) \\
&= \frac{1}{2}h''(u_0)\left(2\frac{1}{4\lambda^2}h''(u_0)\,\eta_1^2\eta_1 + \eta_1\eta_2\right) + \frac{1}{6}h'''(u_0)\left(3\left(\frac{1}{4\lambda^2}h''(u_0)\right)^2\eta_1^4 + 3\frac{1}{4\lambda^2}h''(u_0)\,\eta_1^2\eta_1\eta_2 + \frac{3}{4}\eta_1^3\right) \\
&= \eta_1^3\left(\frac{1}{4\lambda^2}\left(h''(u_0)\right)^2 + \frac{1}{8}h'''(u_0) + \frac{1}{2}h'''(u_0)\left(\frac{1}{4\lambda^2}h''(u_0)\right)^2\eta_1\right) + \eta_1\eta_2\left(\frac{1}{2}h''(u_0) + 3\frac{1}{4\lambda^2}\frac{1}{6}h'''(u_0)\,h''(u_0)\,\eta_1^2\right) \\
&\approx \eta_1^3\left(\frac{1}{4\lambda^2}\left(h''(u_0)\right)^2 + \frac{1}{8}h'''(u_0)\right) + \frac{1}{2}h''(u_0)\,\eta_1\eta_2
\end{aligned}$$

so that

$$\eta_2 = -\left(\frac{1}{2\lambda^2}h''(u_0) + \frac{1}{4}\frac{h'''(u_0)}{h''(u_0)}\right)\eta_1^2$$

From the third equation,

$$
\begin{aligned}
-3\lambda^2\eta_2 &= \frac{1}{2}h''(u_0)\left(2\eta_0\eta_2+\frac{1}{2}\eta_1^2\right)+\frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_2+\frac{3}{2}\eta_0\eta_1^2\right)\\
3\lambda^2\left(\frac{1}{2\lambda^2}h''(u_0)+\frac{1}{4}\frac{h'''(u_0)}{h''(u_0)}\right)\eta_1^2 &= \frac{1}{2}h''(u_0)\left(-2\frac{1}{4\lambda^2}h''(u_0)\left(\frac{1}{2\lambda^2}h''(u_0)+\frac{1}{4}\frac{h'''(u_0)}{h''(u_0)}\right)\eta_1^4+\frac{1}{2}\eta_1^2\right)\\
&\quad+\frac{1}{6}h'''(u_0)\left(-3\left(\frac{1}{4\lambda^2}h''(u_0)\right)^2\left(\frac{1}{2\lambda^2}h''(u_0)+\frac{1}{4}\frac{h'''(u_0)}{h''(u_0)}\right)\eta_1^6+\frac{3}{2}\frac{1}{4\lambda^2}h''(u_0)\eta_1^4\right)\\
h'''(u_0) &= -\frac{5}{3\lambda^2}\left(h''(u_0)\right)^2 \qquad (30)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
-8\lambda^2\eta_3 &= \frac{1}{2}h''(u_0)\left(2\eta_0\eta_3+\eta_1\eta_2\right)+\frac{1}{6}h'''(u_0)\left(3\eta_0^2\eta_3+\frac{1}{4}\eta_1^3+3\eta_0\eta_1\eta_2\right)\\
-8\lambda^2\eta_3 &= \frac{1}{2}h''(u_0)\left(2\frac{1}{4\lambda^2}h''(u_0)\eta_1^2\eta_3-\left(\frac{1}{2\lambda^2}h''(u_0)+\frac{1}{4}\frac{h'''(u_0)}{h''(u_0)}\right)\eta_1^3\right)\\
&\quad+\frac{1}{6}h'''(u_0)\left(3\left(\frac{1}{4\lambda^2}h''(u_0)\right)^2\eta_1^4\eta_3+\frac{1}{4}\eta_1^3-3\frac{1}{4\lambda^2}h''(u_0)\left(\frac{1}{2\lambda^2}h''(u_0)+\frac{1}{4}\frac{h'''(u_0)}{h''(u_0)}\right)\eta_1^2\eta_1^2\eta_1\right)\\
0 &= \eta_3\left(8\lambda^2+\frac{1}{4\lambda^2}\left(h''(u_0)\right)^2\eta_1^2+\frac{1}{2}h'''(u_0)\left(\frac{1}{4\lambda^2}h''(u_0)\right)^2\eta_1^4\right)\\
&\quad-\left(\frac{1}{12}h'''(u_0)+\frac{1}{4\lambda^2}\left(h''(u_0)\right)^2\right)\eta_1^3\\
&\approx \eta_3 8\lambda^2\\
&\quad-\left(\frac{1}{12}h'''(u_0)+\frac{1}{4\lambda^2}\left(h''(u_0)\right)^2\right)\eta_1^3
\end{aligned}
$$

so that

$$
\eta_3=\left(\frac{1}{96\lambda^2}h'''(u_0)+\frac{1}{32\lambda^4}\left(h''(u_0)\right)^2\right)\eta_1^3
$$

Collecting our results,

$$
\begin{aligned}
\eta_0 &= \frac{1}{4\lambda^2}h''(u_0)\eta_1^2\\
\eta_2 &= -\left(\frac{1}{2\lambda^2}h''(u_0)+\frac{1}{4}\frac{h'''(u_0)}{h''(u_0)}\right)\eta_1^2\\
h'''(u_0) &= -\frac{5}{3\lambda^2}\left(h''(u_0)\right)^2\\
\eta_3 &= \left(\frac{1}{96\lambda^2}h'''(u_0)+\frac{1}{32\lambda^4}\left(h''(u_0)\right)^2\right)\eta_1^3
\end{aligned}
$$

If we use the third to eliminate h''',

$$
\begin{aligned}
\eta_0 &= \frac{1}{4\lambda^2}h''(u_0)\eta_1^2\\
\eta_2 &= -\frac{1}{12\lambda^2}h''(u_0)\eta_1^2\\
\eta_3 &= \frac{1}{72\lambda^4}\left(h''(u_0)\right)^2\eta_1^3\\
h'''(u_0) &= -\frac{5}{3\lambda^2}\left(h''(u_0)\right)^2
\end{aligned}
$$

Now, we already know that

$$h = \frac{\mu A}{L_\varphi^2} u^{1-\lambda^2}$$
$$h' = 1 - \lambda^2$$

so taking a derivative and equating,

$$\begin{aligned} h' &= \left(1-\lambda^2\right) \frac{\mu A}{L_\varphi^2} u^{-\lambda^2} \\ 1-\lambda^2 &= \left(1-\lambda^2\right) \frac{\mu A}{L_\varphi^2} u^{-\lambda^2} \\ 1 &= \frac{\mu A}{L_\varphi^2} u^{-\lambda^2} \end{aligned} \tag{31}$$

Then, differentiating further,

$$\begin{aligned} h'' &= -\lambda^2\left(1-\lambda^2\right) \frac{\mu A}{L_\varphi^2} u^{-\lambda^2-1} \\ h''' &= \lambda^2\left(1-\lambda^2\right)\left(\lambda^2+1\right) \frac{\mu A}{L_\varphi^2} u^{-\lambda^2-2} \end{aligned}$$

Evaluating at u_0 and using Eq.(31),

$$\begin{aligned} h''(u_0) &= -\frac{\lambda^2\left(1-\lambda^2\right)}{u_0} \\ h'''(u_0) &= \frac{\lambda^2\left(1-\lambda^2\right)\left(\lambda^2+1\right)}{u_0^2} \end{aligned}$$

Now the equality of derivatives, Eq.(30), becomes

$$\begin{aligned} h'''(u_0) &= -\frac{5}{3\lambda^2}\left(h''(u_0)\right)^2 \\ 3\lambda^2 h'''(u_0) + 5\left(h''(u_0)\right)^2 &= 0 \\ 3\lambda^2 \frac{\lambda^2\left(1-\lambda^2\right)\left(\lambda^2+1\right)}{u_0^2} + 5\frac{\lambda^4\left(1-\lambda^2\right)^2}{u_0^2} &= 0 \\ 3\lambda^4\left(1-\lambda^2\right)\left(\lambda^2+1\right) + 5\lambda^4\left(1-\lambda^2\right)^2 &= 0 \\ \lambda^4\left(1-\lambda^2\right)\left(3\left(\lambda^2+1\right)+5\left(1-\lambda^2\right)\right) &= 0 \\ \lambda^4\left(1-\lambda^2\right)\left(3\lambda^2+3+5-5\lambda^2\right) &= 0 \\ 2\lambda^4\left(1-\lambda^2\right)\left(4-\lambda^2\right) &= 0 \end{aligned}$$

and we see that the only candidate power laws, $f(r) = Ar^{\lambda^2-3}$, for stable, closed orbits are:

$$\begin{aligned} f(r) &= Ar^{-3} \\ f(r) &= Ar^{-2} \\ f(r) &= Ar \end{aligned}$$

The first of these yields only perfectly circular orbits, so the only nontrivial cases are the inverse square law and Hookes' law.

This condition only shows that these power laws are necessary. That they are sufficient to produce closed orbits requires solving for their orbits exactly, which we have already done.

11 Scattering

Before leaving our discussion of central forces, we discuss the extremely important computation of scattering cross-sections. Predictions of these give interpretations of the meaning of collisions in particle accelerators.

11.1 General formalism: Center of Mass Frame

Unbounded orbits are typically encountered in scattering experiments. A beam of particles is directed at a target, and the resulting interactions of the beam with the target particles scatters the beam particles in all directions, with a probability that depends on the forces. The target particles may be molecules, atoms, nuclei, or other fundamental particles, and the beam generally consists of electrons, protons or heavy nuclei.

The variable measured is called the differential cross-section, $d\sigma$, defined as

$$d\sigma = \frac{d\sigma}{d\Omega} d\Omega = \frac{number\ of\ particles\ scattered\ into\ solid\ angle\ d\Omega\ per\ unit\ time}{incident\ intensity\ (particles\ per\ unit\ area\ per\ unit\ time)}$$

with units of area. The solid angle, $d\Omega$, is given by

$$d\Omega = \sin\theta d\Theta d\Phi$$

where our use of upper case Greek letters distinguishes the center of mass frame (Θ, Φ) from the lab frame (θ, φ). We take the z-axis as the direction of the incident beam, so that deviations from that direction are given by Θ or θ. Since scattering by central forces cannot depend on azimuthal angle, we may integrate over φ and look at the probability for scattering into an annulus at angle Θ, of solid angle

$$d\Omega = 2\pi \sin\Theta d\Theta$$

The approaching beam of particles (of intensity, I) is generally moving at a fixed velocity so the total energy, $E = \frac{1}{2}mv_0^2$, of the scattering is the same for all ecounters. It is therefore only the total angular momentum, l, that determines the angle of scatter. We can find l by extending the initial beam particle trajectory past the target. The distance of closest approach of this line is called the impact parameter, s, and the angular momentum about the target is just

$$\begin{aligned} L &= mv_0 s \\ &= s\sqrt{2mE} \end{aligned}$$

Notice that this is different from the actual r_{min} as the beam gets close to the target and is attracted or repelled by it.

If we let $N(\Theta)\, d\Omega$ be the number of particles scattered between Θ and $\Theta + d\Theta$,

$$N(\Theta)\, d\Omega = 2\pi I \left(\frac{d\sigma}{d\Omega}\right) \sin\Theta\, |d\Theta|$$

this must equal the number with impact parameter between the corresponding s and $s + ds$,

$$N(\Theta)\, d\Omega = 2\pi s I\, |ds|$$

where by the corresponding s, we mean the value of the impact parameter leading to a deflection by Θ. We we need only find the relationship between impact parameter and scattering angle,

$$s(\Theta, E)$$

Then we have

$$\begin{aligned} 2\pi I \left(\frac{d\sigma}{d\Omega}\right) \sin\Theta\, |d\Theta| &= 2\pi s I\, |ds| \\ \frac{d\sigma}{d\Omega} &= \frac{s}{\sin\Theta} \left|\frac{ds}{d\Theta}\right| \end{aligned}$$

From our discussion of central forces, we have Eq.(8) for the angle as an integral over $u = \frac{1}{r}$,

$$\begin{aligned}
L &= s\sqrt{2mE} \\
\frac{\sqrt{2m}}{L}(\Psi - \Psi_0) &= -\int_{r_{min}}^{r} \frac{du}{\sqrt{E - \frac{L^2}{2m}u^2 - V\left(\frac{1}{u}\right)}} \\
\frac{1}{s\sqrt{E}}(\Psi - \Psi_0) &= -\int_{r_{min}}^{r} \frac{du}{\sqrt{E}\sqrt{1 - \frac{L^2}{2mE}u^2 - \frac{V}{E}}} \\
\Psi &= -\int_{r_{min}}^{r} \frac{sdu}{\sqrt{1 - s^2u^2 - \frac{V}{E}}}
\end{aligned}$$

where in the last step we have set $\Psi_0 = 0$ when $r = r_{min}$. Then if we integrate out to $r = \infty$, we see that the resulting total deflection, $2\Psi_\infty$ is the complement to Θ,

$$\Theta = \pi - 2\Psi_\infty$$

Thus,

$$\Psi_\infty = -s\int_{r_{min}}^{\infty} \frac{du}{\sqrt{1 - \frac{V(u)}{E} - s^2u^2}}$$

Inverting this gives the desired function, $s(\Theta, E)$.

11.2 Coulomb Scattering

As an important example, consider the case of repulsive Coulomb scattering,

$$\begin{aligned}
f(r) &= \frac{ZZ'e^2}{r^2} \\
&= -\frac{\alpha}{r^2}
\end{aligned}$$

Then we have hyperbolic orbits with $r = r_{min}$ and $\dot{r} = 0$ when $\varphi = 0$,

$$r = \frac{2l^2}{1 + \epsilon\cos\varphi}$$

where $l^2 = \frac{L^2}{2\alpha\mu} = \frac{Es^2}{\alpha}$ and $\epsilon = \frac{2l^2}{r_{min}} - 1 = \frac{2Es^2}{\alpha r_{min}} - 1$. From the initial conditions, the energy is given by

$$\begin{aligned}
E &= \frac{1}{2}mr_{min}^2\dot{\varphi}^2 - \frac{\alpha}{r_{min}} \\
&= \frac{L^2}{2mr_{min}^2} - \frac{\alpha}{r_{min}}
\end{aligned}$$

and we may solve for r_{min},

$$\begin{aligned}
Er_{min}^2 &= \frac{L^2}{2m} - \alpha r_{min} \\
Er_{min}^2 + \alpha r_{min} - \frac{L^2}{2m} &= 0 \\
r_{min} &= \frac{1}{2E}\left(-\alpha + \sqrt{\alpha^2 + \frac{4EL^2}{2m}}\right) \\
&= \frac{\alpha}{2E}\left(-1 + \sqrt{1 + \frac{4EL^2}{2m\alpha^2}}\right) \\
&= \frac{\alpha}{2E}\left(-1 + \sqrt{1 + \frac{4E^2s^2}{\alpha^2}}\right)
\end{aligned}$$

where we choose the + sign so that $r_{min} > 0$. Now using $\epsilon = \frac{2Es^2}{\alpha r_{min}} - 1$,

$$\begin{aligned}
\epsilon &= \frac{2Es^2}{\alpha r_{min}} - 1 \\
(\epsilon + 1) &= \frac{4E^2s^2}{\alpha^2\left(-1 + \sqrt{1 + \frac{4E^2s^2}{\alpha^2}}\right)} \\
-(\epsilon + 1) + (\epsilon + 1)\sqrt{1 + \frac{4E^2s^2}{\alpha^2}} &= \frac{4E^2s^2}{\alpha^2} \\
(\epsilon + 1)\sqrt{1 + \frac{4E^2s^2}{\alpha^2}} &= \frac{4E^2s^2}{\alpha^2} + (\epsilon + 1) \\
(\epsilon + 1)^2\left(1 + \frac{4E^2s^2}{\alpha^2}\right) &= \left(\frac{4E^2s^2}{\alpha^2}\right)^2 + 2(\epsilon + 1)\frac{4E^2s^2}{\alpha^2} + (\epsilon + 1)^2 \\
(\epsilon + 1)^2\frac{4E^2s^2}{\alpha^2} &= \left(\frac{4E^2s^2}{\alpha^2}\right)^2 + 2(\epsilon + 1)\frac{4E^2s^2}{\alpha^2} \\
(\epsilon + 1)^2 &= \left(\frac{4E^2s^2}{\alpha^2}\right) + 2(\epsilon + 1) \\
\epsilon^2 - 1 &= \frac{4E^2s^2}{\alpha^2} \\
\epsilon &= \sqrt{1 + \frac{4E^2s^2}{\alpha^2}}
\end{aligned}$$

This gives the orbit in terms of the energy and the impact parameter:

$$r = \frac{\left(\frac{2Es^2}{\alpha}\right)}{1 + \sqrt{1 + \frac{4E^2s^2}{\alpha^2}}\cos\varphi}$$

Now to find $\Theta(s,E)$ we look at the limiting angle as r diverges. For simplicity, keeping $\epsilon = \sqrt{1+\frac{4E^2s^2}{\alpha^2}}$

$$\begin{aligned}
1+\epsilon\cos\Psi_\infty &= 0 \\
\cos\Psi_\infty &= -\frac{1}{\epsilon} \\
\Psi_\infty &= \cos^{-1}\left(-\frac{1}{\epsilon}\right) \\
&= \cos^{-1}\left(\frac{1}{\epsilon}\right)
\end{aligned}$$

The scattering angle is now

$$\begin{aligned}
\Theta &= \pi - 2\Psi_\infty \\
&= \pi - 2\cos^{-1}\left(\frac{1}{\epsilon}\right) \\
\frac{\Theta}{2} - \frac{\pi}{2} &= \cos^{-1}\left(\frac{1}{\varepsilon}\right) \\
\cos\left(\frac{\Theta}{2} - \frac{\pi}{2}\right) &= \frac{1}{\varepsilon} \\
\cos\left(\frac{\Theta}{2} - \frac{\pi}{2}\right) &= -\sin\left(\frac{\Theta}{2}\right) \\
\sin\frac{\Theta}{2} &= \frac{1}{\varepsilon}
\end{aligned}$$

Now solve for $s(\Theta)$,

$$\begin{aligned}
\varepsilon &= \sqrt{1+\left(\frac{2Es}{ZZ'e^2}\right)^2} \\
\varepsilon^2 - 1 &= \left(\frac{2Es}{ZZ'e^2}\right)^2 \\
s &= \frac{ZZ'e^2}{2E}\sqrt{\varepsilon^2-1}
\end{aligned}$$

where $\epsilon = \frac{1}{\sin\frac{\Theta}{2}}$, so

$$\begin{aligned}
s &= \frac{ZZ'e^2}{2E}\sqrt{\frac{1}{\sin^2\frac{\Theta}{2}} - 1} \\
&= \frac{ZZ'e^2}{2E}\sqrt{\frac{1-\sin^2\frac{\Theta}{2}}{\sin^2\frac{\Theta}{2}}} \\
&= \frac{ZZ'e^2}{2E}\cot\frac{\Theta}{2}
\end{aligned}$$

Now compute the cross-setion,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \\ &= \frac{\left(\frac{ZZ'e^2}{2E}\frac{\cos\frac{\Theta}{2}}{\sin\frac{\Theta}{2}}\right)}{2\cos\frac{\Theta}{2}\sin\frac{\Theta}{2}}\frac{ZZ'e^2}{4E\sin^2\frac{\Theta}{2}} \\ &= \frac{Z^2Z'^2e^4}{16E^2\sin^4\frac{\Theta}{2}}\end{aligned}$$

The result is the *Rutherford cross section*

$$\frac{d\sigma}{d\Omega} = \frac{Z^2Z'^2e^4}{16E^2\sin^4\frac{\Theta}{2}} \tag{32}$$

which allowed for the discovery of the nucleus in the early 20th century, even though the calculation properly requires a quantum treatment. It is a happy coincidence that the full quantum treatment still gives the characteristic $\frac{1}{\sin^4\frac{\Theta}{2}}$ behavior.

11.3 Laboratory frame

The cross-section calculations above were performed in the center of mass frame of reference (CM frame). We now transform to the laboratory frame.

Let the center of mass of our two particle system be $\mathbf{R}$ and have velocity $\mathbf{V}$, where m_1 is the mass of our beam particle and m_2 the mass of the target. Both particles will move after the collision. Denoting laboratory position and velocity by $(\mathbf{r},\mathbf{v})$ and the corresponding center of mass variables as $(\mathbf{r}',\mathbf{v}')$, we have for the incident particle,

$$\mathbf{r}_1 = \mathbf{R}+\mathbf{r}_1' \tag{33}$$

$$\mathbf{v}_1 = \mathbf{V}+\mathbf{v}_1' \tag{34}$$

In the lab frame, m_1 approaches with velocity $\mathbf{v}_0$ with m_2 at rest, giving total momentum $m_1\mathbf{v}_0$. In the CM frame the two particles have equal but opposite momentum, so the only net momentum is the relative motion of the CM frame itself, $(m_1+m_2)\,\mathbf{V}$, so we must have

$$\begin{aligned}(m_1+m_2)\,\mathbf{V} &= m_1\mathbf{v}_0 \\ (m_1+m_2)\,\mathbf{V} &= \frac{m_1m_2}{m_2}\mathbf{v}_0 \\ \mathbf{V} &= \frac{\mu}{m_2}\mathbf{v}_0\end{aligned}$$

with both velocities in the z-direction. Let Θ be the angle of $\mathbf{v}_1'$ with the z-axis, and θ be the angle of $\mathbf{v}_1$ with the z-axis. Then writing the velocity transformation, Eq.(34) in components,

$$\begin{aligned}v_1\sin\theta &= v_1'\sin\Theta \\ v_1\cos\theta &= v_1'\cos\Theta+V\end{aligned}$$

Taking the ratio of these two equations gives the relationship between the center of mass and lab frame angles,

$$\tan\theta = \frac{\sin\Theta}{\cos\Theta+\frac{V}{v_1'}} \tag{35}$$

once we find the ratio $\frac{V}{v_1'}$ between the center of mass velocity and the magnitude of the final velocity of m_1 in the center of mass frame.

Let the relative velocity of the particles after the collision be $\mathbf{v}$. For an elastic collision, we have $v = v_0$, but in general we are interested in cases where some energy goes into the final particles, for example, if we want to produce an excited state of an atom. In general, if the particles absorb an energy $-Q$ with $Q > 0$, we have

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\mu v_0^2 - Q$$

and dividing by $\frac{1}{2}\mu v_0^2$,

$$\begin{aligned}
\frac{v^2}{v_0^2} &= 1 - \frac{Q}{\frac{1}{2}\mu v_0^2} \\
\frac{v}{v_0} &= \sqrt{1 - \frac{m_1 + m_2}{m_2}\frac{Q}{\frac{1}{2}m_1 v_0^2}} \\
\frac{v}{v_0} &= \sqrt{1 - \frac{m_1 + m_2}{m_2}\frac{Q}{E}}
\end{aligned}$$

where E the incident lab frame energy, $E = \frac{1}{2}m_1 v_0^2$. From Eq.(4) the center of mass velocity is given by

$$\mathbf{v}_1' = \dot{\mathbf{x}}_{m_1} - \dot{\mathbf{R}}_{cm} = \frac{M\dot{\mathbf{r}}}{M+m} = \frac{M\mathbf{v}}{M+m}$$

with magnitude

$$v_1' = \frac{Mv}{M+m} = \frac{\mu}{m_1}v$$

so

$$\begin{aligned}
\frac{V}{v_1'} &= \frac{\frac{\mu}{m_2}v_0}{\frac{\mu}{m_1}v} \\
&= \frac{m_1}{m_2}\cdot\frac{v_0}{v} \\
&= \frac{m_1}{m_2\sqrt{1 - \frac{m_1+m_2}{m_2}\frac{Q}{E}}}
\end{aligned}$$

Substituting into Eq.(35) completes the relationship between the angles and the energies,

$$\tan\theta = \frac{\sin\Theta}{\cos\Theta + \frac{m_1}{m_2\sqrt{1-\frac{m_1+m_2}{m_2}\frac{Q}{E}}}} \tag{36}$$

Generally, we are interested in maximizing the energy Q available for creating new excited states. In terms of the incident lab frame velocity, v_0, the initial energy is $\frac{1}{2}m_1 v_0^2$, while the minimum final energy occurs when both final particles remain together at the center of mass, in a totally inelastic collision. In a totally inelastic collision, the $v = 0$ and there is a single combined final velocity v_f. Since we must conserve momentum, the final momentum will be

$$\begin{aligned}
(m_1 + m_2)\, v_f &= m_1 v_0 \\
v_f &= \frac{m_1}{m_1 + m_2} v_0
\end{aligned}$$

and the change in energy is

$$\begin{aligned}
Q &= \frac{1}{2}m_1v_0^2 - \frac{1}{2}(m_1+m_2)v_f^2 \\
&= \frac{1}{2}m_1v_0^2 - \frac{1}{2}\frac{m_1^2}{m_1+m_2}v_0^2 \\
&= \frac{1}{2}\left(\frac{m_1(m_1+m_2)-m_1^2}{m_1+m_2}\right)v_0^2 \\
&= \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)v_0^2 \\
&= \frac{m_2}{m_1+m_2}E
\end{aligned}$$

If the laboratory frame is the same as the center of mass frame then the final velocity v_f in a totally inelastic collision is zero, so that

$$Q = \frac{1}{2}m_1v_0^2 = E$$

and the entire initial energy is available for creating excited states. If m_1 and m_2 are comparable this is a substantial difference, as much as twice the energy. This is the underlying principle of a colliding synchrotron, where protons are collided into one another.

11.3.1 Relativistic scattering

This picture is radically altered with a relativistic treatment. The change in energy for an inelastic collision is now given by

$$Q = \left[\frac{m_1c^2}{\sqrt{1-\frac{v_0^2}{c^2}}} + m_2c^2\right] - \left[\frac{(m_1+m_2)c^2}{\sqrt{1-\frac{v_f^2}{c^2}}}\right]$$

where conservation of momentum is

$$\frac{m_1v_0}{\sqrt{1-\frac{v_0^2}{c^2}}} = \frac{(m_1+m_2)v_f}{\sqrt{1-\frac{v_f^2}{c^2}}}$$

Solving for $\sqrt{1-\frac{v_f^2}{c^2}}$,

$$\begin{aligned}
m_1v_0\sqrt{1-\frac{v_f^2}{c^2}} &= (m_1+m_2)v_f\sqrt{1-\frac{v_0^2}{c^2}} \\
\frac{m_1v_0}{(m_1+m_2)\sqrt{1-\frac{v_0^2}{c^2}}}\sqrt{1-\frac{v_f^2}{c^2}} &= v_f \\
\frac{m_1^2v_0^2}{(m_1+m_2)^2\left(1-\frac{v_0^2}{c^2}\right)}\left(1-\frac{v_f^2}{c^2}\right) &= v_f^2 \\
1-\frac{m_1^2v_0^2}{(m_1+m_2)^2c^2\left(1-\frac{v_0^2}{c^2}\right)}\left(1-\frac{v_f^2}{c^2}\right) &= 1-\frac{v_f^2}{c^2}
\end{aligned}$$

Then, solving for $\sqrt{1-\frac{v_f^2}{c^2}}$,

$$
\begin{aligned}
1 &= \left(1+\frac{m_1^2 v_0^2}{\left(m_1+m_2\right)^2 c^2\left(1-\frac{v_0^2}{c^2}\right)}\right)\left(1-\frac{v_f^2}{c^2}\right) \\
\sqrt{1-\frac{v_f^2}{c^2}} &= \sqrt{\frac{1}{1+\frac{m_1^2 v_0^2}{\left(m_1+m_2\right)^2 c^2\left(1-\frac{v_0^2}{c^2}\right)}}} \\
&= \sqrt{\frac{\left(m_1+m_2\right)^2 c^2\left(1-\frac{v_0^2}{c^2}\right)}{\left(m_1+m_2\right)^2 c^2\left(1-\frac{v_0^2}{c^2}\right)+m_1^2 v_0^2}} \\
&= \frac{\left(m_1+m_2\right) c\sqrt{1-\frac{v_0^2}{c^2}}}{\sqrt{\left(m_1+m_2\right)^2 c^2\left(1-\frac{v_0^2}{c^2}\right)+m_1^2 v_0^2}}
\end{aligned}
$$

and therefore the available energy is

$$
\begin{aligned}
Q &= \frac{m_1 c^2}{\sqrt{1-\frac{v_0^2}{c^2}}}+m_2 c^2-\frac{\left(m_1+m_2\right) c^2}{\sqrt{1-\frac{v_f^2}{c^2}}} \\
&= \frac{m_1 c^2}{\sqrt{1-\frac{v_0^2}{c^2}}}+m_2 c^2-\frac{c}{\sqrt{1-\frac{v_0^2}{c^2}}}\sqrt{\left(m_1+m_2\right)^2 c^2\left(1-\frac{v_0^2}{c^2}\right)+m_1^2 v_0^2} \\
&= \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}\left[m_1 c^2+m_2 c^2\sqrt{1-\frac{v_0^2}{c^2}}-c\sqrt{\left(m_1+m_2\right)^2 c^2\left(1-\frac{v_0^2}{c^2}\right)+m_1^2 v_0^2}\right]
\end{aligned}
$$

If the speed is close to the speed of light so that $1-\frac{v_0^2}{c^2}\approx 0$, this is approximately

$$
\begin{aligned}
Q &\approx \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}\left[m_1 c^2-c\sqrt{m_1^2 v_0^2}\right] \\
&= \frac{m_1 c^2}{\sqrt{1-\frac{v_0^2}{c^2}}}\left(1-\frac{v_0}{c}\right) \\
&= E_1\left(1-\frac{v_0}{c}\right)\approx 0
\end{aligned}
$$

which is a very small fraction of the energy of the incident particle. This makes it crucial that modern synchrotrons are colliding beam accelerators, in which particles are accelerated in opposite directions. The collision then takes place in such a way that the center of mass and lab frames are the same, and we have $Q_{max}=E$.

11.4 Quasi-stable states

For more complicated potentials, where both attractive and repulsive forces act, there may be a potential well: with increasing radius, the potential may fall to a minimum, rise to a local maxiumum, then drop off again. Classically, any particle entering such a well will pass inside it until it hits a turning point, then emerge again. The only interesting case is when the energy is right near the top of the local maxiumum. In this case, the radial motion becomes very slow and the beam particle may orbit the center multiple times before escaping a near-circular orbit at the local maximum.

This is not, in fact, what usually happens. When the problem is treated quantum mechanically, it is possible for the beam particle to tunnel into the well even if its energy is below the local maximum. When this happens, there is a classically bound state formed, with the beam particle trapped in the well. Since the system is quantum mechanical, the orbiting particle will eventually tunnel out again, with the duration of the state determined by the tunneling probability. Such a state is called quasi-stable. While very important in particle physics, these states take us beyond the concerns of classical physics.