Chapter 1

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4.1 Constraints

We are often interested in problems which do not allow all particles a full range of motion, but instead restrict motion to some subspace. When constrained motion can be described in this way, there is a simple technique for formulating the problem.

Subspaces of constraint may be described by one or more relationships between the coordinates,

$$f_m(\mathbf{x}, t) = 0$$

where $m = 1, 2, \cdots, k$ if there are k different constraints. The trick is to introduce the constraints $f_m = 0$ into the problem in such a way that it must vanish in the solution. Our understanding of the Euler-Lagrange equation as the covariant form of Newton's second law tells us how to do this.

First consider a free particle, with Lagrangian $L = \frac{1}{2}m\dot{\mathbf{x}}^2$. The Euler-Lagrange equation is then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{dp_i}{dt} = 0$$

Now let $f(\mathbf{x}, t) = 0$ be a constraint on the motion of this particle. The constraint equation describes a surface limiting the motion and must do so by applying a force on the particle in a direction perpendicular to the surface. This is the same direction as the gradient of f, so we must be able to write the force of constraint as

$$\mathbf{F}_{constraint} = \lambda \boldsymbol{\nabla} f(\mathbf{x}, t)$$

or if there is more than one constraint,

$$\mathbf{F}_{constraint} = \sum_{m=1}^{k} \lambda_m \boldsymbol{\nabla} f_m(\mathbf{x}, t)$$

Then the equation of motion for the otherwise free particle would be

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{dp_i}{dt} = \lambda \boldsymbol{\nabla} f(\mathbf{x}, t)$$

We may arrive at this equation by adding λf to the Lagrangian, since with $L' = \frac{1}{2}m\dot{\mathbf{x}}^2 + \lambda f((\mathbf{x}, t))$ the Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}^i} - \frac{\partial L'}{\partial x^i} = 0$ becomes

$$\frac{d\mathbf{p}}{dt} = \lambda \boldsymbol{\nabla} f(\mathbf{x}, t)$$

The only thing that remains is to impose the constraint, but this may also be accomplished by variation if we regard λ as an independent variable. Then, varying λ in the action we have

$$\begin{aligned} S &= \int \left(\frac{1}{2}m\dot{\mathbf{x}}^2 + \lambda f\right) dt \\ 0 &= \delta_\lambda S \\ &= \int \delta\lambda f dt \end{aligned}$$

which holds for arbitrary variations if and only if $f(\mathbf{x}, t) = 0$. The additional degree of freedom, λ, is called a *Lagrange multiplier*.

Notice that the term we have added to the Euler-Lagrange equation transforms covariantly when we change coordinates, since

$$\left(\frac{\partial f}{\partial q^i}\right) = \frac{\partial x^j}{\partial q^i}\left(\frac{\partial f}{\partial x^k}\right)$$

as required for general coordinate covariance of the equation.

All of this is perfectly legal, since we are forcing $f(\mathbf{x}, t) = 0$. This means that the term λf that we add to the action is actually zero.

Generalizing, to arbitrarily many constraints and arbitrary coordinates, $f_m(\mathbf{q}, t) = 0$ we increase the number of independent coordinates by k Lagrange multipliers, λ_m, while the variation of the multipliers decreases the number of independent coordinates by k constraints $f_m = 0$. The total vanishing product

$$\sum_{m=1}^{k} \lambda_m f_m(\mathbf{q}, t) = 0$$

is added to the Lagrangian and the action becomes

$$S = \int \left(L + \sum \lambda_m f_m\right) dt$$

The resulting set of equations after varying the coordinates and the λ_m are:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} &= \sum_{m=1}^{k} \lambda_m \frac{\partial f_m}{\partial q^i} \\ f_m(\mathbf{q}, t) &= 0 \end{aligned}$$

We now have $n + k$ equations, and k of them are the constraints.

These are exactly what we require – the extra equation gives just enough information to determine λ, while the addition to the Euler-Lagrange equation is the force of constraint.

Thus, by increasing the number of degrees of freedom of the problem by one for each constraint, we include the constraint while allowing free variation of the action. In exchange for the added equation of motion, we learn that the force required to maintain the constraint is

$$\mathbf{F}_{constraint} = \lambda \boldsymbol{\nabla} f$$

The advantage of treating constraints in this way is that we now may carry out the variation of the coordinates freely, as if all motions were possible. The variation of the Lagrange multipliers λ_m brings in the constraint automatically. In the end, we will have the $n - k$ Euler-Lagrange equations for the evolution of the coordinates, plus an additional equation that determines each Lagrange multiplier.

When the constraint surface is fixed in space the constraint force never does any work since there is never any motion in the direction of the gradient of f. If there is time dependence of the surface then work will be done. Because each f_m remains zero its total time derivative vanishes so

$$
\begin{aligned}
0 &= \frac{df_m}{dt} \\
&= \boldsymbol{\nabla} f_m \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f_m}{\partial t}
\end{aligned}
$$

for each m. Multiplying by $\lambda_m dt$ and integrating (but not summing),

$$
\int \lambda_m \boldsymbol{\nabla} f_m \cdot d\mathbf{x} = -\int \lambda_m \frac{\partial f_m}{\partial t}
$$

The expression on the left is just the line integral of the force of constraint, that is, the work:

$$
\int \lambda_m \boldsymbol{\nabla} f_m \cdot d\mathbf{x} = \int \mathbf{F}_{constraint} \cdot d\mathbf{x} = W
$$

and therefore the work done by the moving constraint is

$$
W = \int \mathbf{F}_{constraint} \cdot d\mathbf{x} = -\int \lambda \frac{\partial f}{\partial t} dt
$$

Thus, the Lagrange multiplier allows us to compute the work done by a moving constraint surface. Notice that if the constraint is independent of time, it does no work.

4.2 Examples

4.2.1 Particle on an inclined plane

As a simple example, consider the motion of a particle under the influence of gravity, $V = mgz$, constrained to a fixed plane inclined at an angle θ. Let the plane slope up to the right so that the angled surface satisfies $\tan\theta = \frac{z}{x}$, so

$$
f(x,z) = z - x\tan\theta = 0
$$

where θ is a fixed angle. We write the action as

$$
S = \int \left(\frac{1}{2} m\dot{\mathbf{x}}^2 - mgz + \lambda\left(z - x\tan\theta\right) \right) dt
$$

Because y is cyclic we immediately have

$$
p_y = m\dot{y} = mv_{0y} = const.
$$

so that

$$
y = y_0 + v_{0y} t
$$

Because $\frac{\partial L}{\partial t} = 0$, we also have conservation of energy,

$$
\begin{aligned}
E &= \sum_i \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \\
&= \frac{1}{2} m\dot{\mathbf{x}}^2 + mgz - \lambda\left(z - x\tan\theta\right)
\end{aligned}
$$

Varying x, z and λ we have three further equations,

$$
\begin{aligned}
0 &= m\ddot{x} + \lambda\tan\theta \\
0 &= m\ddot{z} + mg - \lambda \\
0 &= z - x\tan\theta
\end{aligned}
$$

Using the constraint to simplify z in the conservation of energy, the conservation law reduces to

$$\begin{aligned} E &= \frac{1}{2}m\dot{\mathbf{x}}^2 + mgz - \lambda\left(z - x\tan\theta\right) \\ &= \frac{1}{2}m\dot{\mathbf{x}}^2 + mgz \end{aligned}$$

This shows that for this example the constraint contributes no energy.

Now rewrite the first two equations of motion as

$$\ddot{x} = -\frac{\lambda}{m}\tan\theta \tag{4.1}$$

$$\ddot{z} = -g + \frac{\lambda}{m} \tag{4.2}$$

and differentiate the constraint equation twice to get

$$\ddot{z} - \ddot{x}\tan\theta = 0$$

Substituting for the components of the acceleration, we may solve for the Lagrange multiplier,

$$\begin{aligned} \left(-g + \frac{\lambda}{m}\right) - \left(-\frac{\lambda}{m}\tan\theta\right)\tan\theta &= 0 \\ -g + \frac{\lambda}{m}\left(1 + \tan^2\theta\right) &= 0 \\ \lambda &= \frac{mg}{1+\tan^2\theta} \\ &= mg\cos^2\theta \end{aligned}$$

In this case, λ is constant.

Eliminating λ from Eqs.(4.1) and (4.2), gives two differential equations,

$$\begin{aligned} \ddot{x} &= -g\cos^2\theta\tan\theta \\ &= -g\cos\theta\sin\theta \\ &= -\frac{1}{2}g\sin 2\theta \\ \ddot{z} &= -g + g\cos^2\theta \\ &= -g\sin^2\theta \end{aligned}$$

Check that the magnitude of the acceleration is

$$\begin{aligned} a &= \sqrt{\ddot{x}^2 + \ddot{z}^2} \\ &= \sqrt{g^2\cos^2\theta\sin^2\theta + g^2\sin^4\theta} \\ &= g\sin\theta \end{aligned}$$

as expected. The accelerations are constant and immediately integrated to give

$$\begin{aligned} x &= x_0 + v_{0x}t - \frac{1}{2}\sin 2\theta g t^2 \\ z &= z_0 + v_{0z}t - \frac{1}{2}\sin^2\theta g t^2 \end{aligned}$$

The added benefit of the method is that we have also found the normal force, the force of constraint. It is given by

$$\begin{aligned} \mathbf{F} &= \lambda\boldsymbol{\nabla} f \\ &= mg\cos^2\theta\boldsymbol{\nabla}\left(z - x\tan\theta\right) \\ &= mg\cos\theta\left(-\sin\theta, 0, \cos\theta\right) \end{aligned}$$

Notice that the magnitude is the normal force, $mg\cos\theta$ as expected, with the unit vector $\hat{\mathbf{n}} = (-\sin\theta, 0, \cos\theta)$ correctly giving the normal to the plane. So, while it is true that we could have used the constraint $z = x\tan\theta$ to eliminate z in the original action to write

$$S = \int \left(\frac{1}{2}m\dot{x}^2\left(1+\tan^2\theta\right) - mgx\tan\theta\right) dt$$

and varied only x to find a single equation, $m\ddot{x}\left(1+\tan^2\theta\right) = mg\tan\theta$, we would still have had work to do to find z and the constraint force. In systems where the constraint equation is difficult or impossible to solve for one of the coordinates, the method has strong advantages. Suppose, for example, we have a particle moving on a surface given by $z\sin z - c = 0$. This is a transcendental equation with no solution for $z$in terms of elementary functions.

It is instructive to work the inclined plane problem in the usual adapted coordinates, taking x along the plane up from the bottom and y in the direction normal. Now it is important to write the unconstrained Lagrangian for *general* motions of the block. The kinetic energy is still $T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)$ but to write the vertical height we now need $h = x\sin\theta + y\cos\theta$, giving the unconstrained action for a particle moving in 2-dimensions acted on by gravity,

$$S = \int \left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mg\left(x\sin\theta + y\cos\theta\right)\right) dt$$

The constraint is now simply $y = 0$, so the constrained action is

$$S = \int \left(\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mg\left(x\sin\theta + y\cos\theta\right) + \lambda y\right) dt$$

Varying x, y and λ, the equations of motion are

$$\begin{aligned} -m\ddot{x} - mg\sin\theta &= 0 \\ -m\ddot{y} - mg\cos\theta + \lambda &= 0 \\ y &= 0 \end{aligned}$$

Putting the constraint $y = 0$ into the y equation of motion, we find λ,

$$\lambda = mg\cos\theta$$

so the constraint correctly gives the normal force,

$$\begin{aligned} \mathbf{N} &= \lambda\boldsymbol{\nabla} y \\ &= mg\cos\theta\hat{\mathbf{n}} \end{aligned}$$

where is normal to the incline. For x we immediately integrate,

$$x = x_0 + \dot{x}_0 t - \frac{1}{2}gt^2\sin\theta$$

4.2.2 Orbiting ball and weight

A ball of mass m slides on a horizontal, frictionless tabletop. It is connected by a light cord of length L that runs through a hole in the tabletop to a second mass M that hangs vertically. Find the motion.

One approach which minimizes the task of imposing the constraints is to write the free Lagrangian for both particles,

$$S_0 = \int \left(\frac{1}{2}m\left(\dot{\rho}^2+\rho^2\dot{\varphi}^2\right) + \frac{1}{2}M\dot{z}^2 - Mgz\right) dt$$

where the origin is at the hole and z is measured positive downward. In writing it this way, we have already incorporated the constraint of m to the tabletop and of M to a vertical line. Finally, we need to constrain the length of the cord,

$$z + \rho = L$$

The constrained action is then

$$S_0 = \int \left(\frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 \right) + \frac{1}{2} M \dot{z}^2 - Mgz + \lambda \left(z + \rho - L \right) \right) dt$$

Since φ is cyclic, we immediately have the conjugate momentum conserved,

$$p_\varphi = m\rho^2 \dot{\varphi}$$

The energy is also conserved, so

$$E = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 \right) + \frac{1}{2} M \dot{z}^2 + Mgz - \lambda \left(z + \rho - L \right)$$

Now, varying ρ, z_2 and λ,

$$\begin{aligned} -m\ddot{\rho} + m\rho\dot{\varphi}^2 + \lambda &= 0 \\ -M\ddot{z} - Mg + \lambda &= 0 \\ z + \rho - L &= 0 \end{aligned}$$

Differentiating the constraint, we find

$$\ddot{z} = -\ddot{\rho}$$

so the remaining two equations may be written as

$$\begin{aligned} -m\ddot{\rho} + m\rho\dot{\varphi}^2 + \lambda &= 0 \\ M\ddot{\rho} - Mg + \lambda &= 0 \end{aligned}$$

If we multiply the first by M and the second by m and add, the acceleration drops out, and we may solve for the Lagrange multiplier,

$$\begin{aligned} -Mm\ddot{\rho} + Mm\rho\dot{\varphi}^2 + M\lambda + mM\ddot{\rho} - mMg + m\lambda &= 0 \\ Mm\rho\dot{\varphi}^2 - mMg + (M+m)\lambda &= 0 \end{aligned}$$

and therefore

$$\lambda = \frac{Mm}{M+m} \left(g - \rho\dot{\varphi}^2 \right)$$

Defining the *reduced mass*, $\mu \equiv \frac{Mm}{M+m}$ and substituting this into *either* of the two equations (check this!) gives

$$\ddot{\rho} - \frac{M}{M+m} g - \frac{m}{M+m} \rho\dot{\varphi}^2 = 0$$

Now eliminate $\dot{\varphi}$ using the conservation law,

$$\begin{aligned} \ddot{\rho} &= \frac{M}{M+m} g + \frac{m}{M+m} \rho \left(\frac{p_\varphi}{m\rho^2} \right)^2 \\ &= \frac{M}{M+m} g + \frac{p_\varphi^2}{m(M+m)} \frac{1}{\rho^3} \end{aligned}$$

Multiplying by $\dot{\rho} = \frac{d\rho}{dt}$ should give the energy. We could also find this expression by substituting the results so far into E above. Taking the integration route,

$$\begin{aligned}
\dot{\rho}\frac{d\dot{\rho}}{dt} &= \left(\frac{M}{M+m}g + \frac{p_\varphi^2}{m(M+m)}\frac{1}{\rho^3}\right)\frac{d\rho}{dt} \\
\int\limits_{\dot{\rho}_0}^{\dot{\rho}} \dot{\rho} d\dot{\rho} &= \int\limits_{\rho_0}^{\rho} \left(\frac{M}{M+m}g + \frac{p_\varphi^2}{m(M+m)}\frac{1}{\rho^3}\right) d\rho \\
\frac{1}{2}\dot{\rho}^2 - \frac{1}{2}\dot{\rho}_0^2 &= \frac{M}{M+m}g\rho - \frac{p_\varphi^2}{2m(M+m)}\frac{1}{\rho^2} + \frac{p_\varphi^2}{2m(M+m)}\frac{1}{\rho_0^2}
\end{aligned}$$

Solving for $\dot{\rho} = \frac{d\rho}{dt}$ and writing another integral we have reduced the problem to quadratures,

$$t = \int\limits_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{\frac{2M}{M+m}g\rho - \frac{p_\varphi^2}{m(M+m)}\frac{1}{\rho^2} + \frac{p_\varphi^2}{m(M+m)}\frac{1}{\rho_0^2} + \dot{\rho}_0^2}}$$

This is integrable, but is probably best left to an online integrator, special cases and perturbation, or computer solutions with plotting.

Sometimes (not here) it is easier to write S_0 as completely unconstrained,

$$S_0 = \int \left(\frac{1}{2}m\dot{\mathbf{x}}_1^2 + \frac{1}{2}M\dot{\mathbf{x}}_2^2 + Mgz_2\right) dt$$

Now there are several constraints. To keep m on the plane, we need

$$z_1 = 0$$

while to keep M in a vertical line we set

$$\begin{aligned}
x_2 &= 0 \\
y_2 &= 0
\end{aligned}$$

The resulting constrained action is

$$S = \int \left(\frac{1}{2}m\dot{\mathbf{x}}_1^2 + \frac{1}{2}M\dot{\mathbf{x}}_2^2 - Mgz_2 + \lambda_1 z_1 + \lambda_2 x_2 + \lambda_3 y_2 + \lambda_4\left(-z_2 + \sqrt{\mathbf{x}_1^2} - L\right)\right) dt$$

While the position constraints are too simple to take this approach here, more complicated constraints on the position make this technique useful.

4.2.3 The Brachistochrone I: Roller coaster

Consider the first great hill and dip of a roller coaster. We would like to optimize its shape, from the beginning of the drop at point A to the top of the next hill at B in such a way that the time required for the trip from A to B is a minimum. The only force that does any work is the conservative gravitational force, $-mg\hat{\mathbf{k}}$, but in addition there are unknown forces of constraint holding the car to the track. This is an example of the brachistochrone problem, which played an important role in the development of variational methods.

4.2.3.1 The action and conserved quantities

Let a car of mass m move along the track of height h from rest without friction. If we choose z positive downward, we may take the top of the track as the origin. Then the coordinates of the initial and final points are

$$\begin{aligned}\mathbf{x}_A &= (0,0,0)\\ \mathbf{x}_B &= (L,0,h)\end{aligned}$$

It is clear that the motion stays in the xz-plane because y is cyclic and the initial motion has $y = 0 = \dot{y}$. Regardless of the initial velocity, movement in the y-direction cannot make the car go faster, but always adds distance, thereby increasing the time.

The potential is $V = -mgz$ and the kinetic energy is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right)$ so without the constraint the action is just

$$S = \int_0^{t_f} \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) + mgz\right] dt$$

Since x is cyclic, we have

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

constant. Clearly this is not the case for the constrained motion, however. Though it starts with zero velocity, the car ultimately moves to the right, the constraint forces having accelerated it in that direction.

On the other hand, the forces of constraint do no work, and $\frac{\partial L}{\partial t} = 0$, so we expect energy to be conserved,

$$\begin{aligned}E &= \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{z}}\dot{z} - L\\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) - mgz\end{aligned}$$

The enercy is constant, so we may evaluate E at the initial point, where $\mathbf{x} = 0, \mathbf{v} = 0$ so that the value of the constant is $E = 0$. Then the magnitude of the velocity, $v = \sqrt{\dot{x}^2 + \dot{z}^2}$, is given by

$$v = \sqrt{2gz}$$

However, we do not have either the path or the force of constraint, so we can go no further with the Lagrangian. Even though the energy is only guaranteed to be conserved on the unconstrained path, here it is conserved because the force of constraint, being perpendicular to the motion, does no work.

4.2.3.2 The time functional

Instead of the action, we write the time directly. Integrating $v = \frac{ds}{dt}$ where ds is an infinitesimal section of the track, we have

$$t_f = \int_0^{s_f} \frac{ds}{v}$$

where we may write ds as

$$ds = \sqrt{dx^2 + dz^2}$$

and s_f is the total length of the track. If we parameterize the path by

$$C(\lambda) = (x(\lambda), z(\lambda))$$

then we have

$$
\begin{aligned}
t_f &= \int_0^{s_f} \frac{ds}{v} \\
&= \int_0^{s_f} \frac{\sqrt{dx^2 + dz^2}}{\sqrt{2gz}} \\
&= \int_0^{s_f} \frac{\sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2}}{\sqrt{2gz}} \, d\lambda
\end{aligned}
$$

Thus, t_f is a functional of $x(\lambda)$ and $z(\lambda)$. We may choose λ to be any parameter which is monotonic along the path. Assuming there are no loops in the path, we may let $\lambda = x$, thinking of $z = z(x)$ as the curve. Then

$$
t_f = \int_0^{s_f} \frac{\sqrt{1 + \left(\frac{dz}{dx}\right)^2}}{\sqrt{2gz}} \, dx
$$

Let the dot denote differentiation with respect to x, i.e., $\dot{z} = \frac{dz}{dx}$. Then

$$
t_f[z(x)] = \int_0^{s_f} \sqrt{\frac{1 + \dot{z}^2}{2gz}} \, dx
$$

4.2.3.3 Variation

Since the integrand is a function of z and $\dot{z}$, the extremum is given by the Euler-Lagrange equation,

$$
\begin{aligned}
0 &= \frac{d}{dx}\left(\frac{\partial}{\partial \dot{z}} \sqrt{\frac{1 + \dot{z}^2}{2gz}}\right) - \frac{\partial}{\partial z} \sqrt{\frac{1 + \dot{z}^2}{2gz}} \\
&= \frac{d}{dx}\left(\frac{1}{2} \frac{1}{\sqrt{\frac{1+\dot{z}^2}{2gz}}} \frac{2\dot{z}}{2gz}\right) - \frac{1}{2} \frac{1}{\sqrt{\frac{1+\dot{z}^2}{2gz}}} \left(-\frac{1 + \dot{z}^2}{2gz^2}\right) \\
&= \frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{2gz}\sqrt{1 + \dot{z}^2}}\right) + \frac{1}{2z^{3/2}} \sqrt{\frac{1 + \dot{z}^2}{2g}}
\end{aligned}
$$

Now we resort to a neat trick. Rather than differentiating the complicated expression in the first term, we try to integrate directly by writing the equation in terms of that complicated expression. Multiplying and dividing the second term by $\frac{\sqrt{2gz}}{\dot{z}}$,

$$
\begin{aligned}
0 &= \frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right) + \frac{1}{2}\sqrt{1 + \dot{z}^2} \frac{1}{\sqrt{2g}z^{3/2}} \frac{\left(\frac{\sqrt{2gz}}{\dot{z}}\right)}{\left(\frac{\sqrt{2gz}}{\dot{z}}\right)} \\
&= \frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right) + \frac{1}{2}\left(\frac{\sqrt{2gz}\sqrt{1 + \dot{z}^2}}{\dot{z}}\right) \frac{1}{\sqrt{2g}z^{3/2}} \frac{\dot{z}}{\sqrt{2gz}}
\end{aligned}
$$

we multiply by the ugly factor,

$$
0 = \left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right) \frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{1 + \dot{z}^2}\sqrt{2gz}}\right) + \frac{\dot{z}}{4gz^2}
$$

Then, bringing one term to the other side, we integrate:

$$\int\left(\frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}}\right)d\left(\frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}}\right) = -\int\frac{dz}{4gz^2}$$

The integral on the left is of the form $\int f df = \frac{1}{2}f^2$, so integrating and simplifying,

$$\begin{aligned}\frac{1}{2}\left(\frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}}\right)^2 &= \frac{1}{4gz}-C\\ \frac{\dot{z}}{\sqrt{1+\dot{z}^2}\sqrt{2gz}} &= \sqrt{\frac{1}{2gz}-2C}\\ \frac{\dot{z}}{\sqrt{1+\dot{z}^2}} &= \sqrt{1-4Cgz}\end{aligned}$$

Now let $a\equiv 4Cg$ and solve for $\dot{z}$,

$$\begin{aligned}\dot{z}^2 &= (1-az)\left(1+\dot{z}^2\right)\\ \dot{z}^2\left(1-(1-4Cgz)\right) &= 1-az\\ \dot{z}=\frac{dz}{dx} &= \sqrt{\frac{1-az}{az}}\end{aligned}$$

leading to the integral

$$\begin{aligned}\int_0^x dx &= \int_0^z dz\sqrt{\frac{az}{1-az}}\\ ax &= \int_0^z adz\sqrt{\frac{az}{1-az}}\\ &= \int_0^z\sqrt{\frac{w}{1-w}}dw\end{aligned}$$

where we have set $w=az$. Of course, we may google Wolfram integrator, but it is possible to do this on a napkin as well. Let $w=\sin^2\chi$ so that $dw=2\sin\chi\cos\chi d\chi$ and the integral becomes

$$\begin{aligned}\int_0^z\sqrt{\frac{w}{1-w}}dw &= \int_0^z\sqrt{\frac{\sin^2\chi}{1-\sin^2\chi}}2\sin\chi\cos\chi d\chi\\ &= \int_0^z 2\sin^2\chi d\chi\\ &= \int_0^z\left(\sin^2\chi+1-\cos^2\chi\right)d\chi\\ &= \int_0^z\left(1-\cos 2\chi\right)d\chi\\ &= \left[\chi-\frac{1}{2}\sin 2\chi\right]_0^z\end{aligned}$$

where $\chi = \arcsin\sqrt{az}$. Therefore, writing $\sin 2\chi$ in terms of $\sin\chi$,

$$\begin{aligned}\left[\chi - \frac{1}{2}\sin 2\chi\right]_0^z &= \left[\chi - \sin\chi\sqrt{1-\sin^2\chi}\right]_0^z \\ &= \left[\arcsin\sqrt{az} - \sin\left(\arcsin\sqrt{az}\right)\sqrt{1-\sin^2\left(\arcsin\sqrt{az}\right)}\right]_0^z \\ &= \arcsin\sqrt{az} - \sqrt{az}\sqrt{1-az}\end{aligned}$$

and we have

$$ax = \arcsin\sqrt{az} - \sqrt{az}\sqrt{1-az}$$

This is most easily understood if we write it parametrically. A convenient substitution turns out to be $az = \sin^2\frac{\phi}{2}$. For z running from 0 to $h = \frac{1}{a}$ we then have ϕ running from 0 to π. Then,

$$\begin{aligned}ax &= \arcsin\sqrt{az} - \sqrt{az}\sqrt{1-az} \\ &= \arcsin\sin\frac{\phi}{2} - \sin\frac{\phi}{2}\sqrt{1-\sin^2\frac{\phi}{2}} \\ &= \frac{\phi}{2} - \sin\frac{\phi}{2}\cos\frac{\phi}{2} \\ &= \frac{1}{2}\left(\phi - \sin\phi\right)\end{aligned}$$

while for z we have

$$\begin{aligned}az &= \sin^2\frac{\phi}{2} \\ &= \frac{1}{2}\left(\sin^2\frac{\phi}{2} + 1 - \cos^2\frac{\phi}{2}\right) \\ &= \frac{1}{2}\left(1-\cos\phi\right)\end{aligned}$$

These satisfy the initial condition. Imposing the final condition that $z = h$ so that at $\phi = \pi$ we require $\frac{1}{a} = h$, we find the final parametric equations,

$$\begin{aligned}x &= \frac{h}{2}\left(\phi - \sin\phi\right) \\ z &= \frac{h}{2}\left(1-\cos\phi\right)\end{aligned}$$

These are the equations for a cycloid, the shape a point on a rolling disk traces as the disk moves.

4.3 Hanging Chain

Let a chain of length L and mass M hangs between two points, $\mathbf{x}_1 = (x_1, h_1)$ and $\mathbf{x}_2 = (x_2, h_2)$ where $|x_2 - x_1| < L$. What shape does the chain make?

Let the curve described by the chain be parameterized by the height, $y(x)$ at any horizontal position, x. Choose the origin below the lowest point of the chain, $\mathbf{x} = (0, y_{min})$.

Consider an element of the chain of length $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx$. The linear density of the chain is $\rho = \frac{M}{L}$ so the mass of the element is ρds. The chain is not moving we may set the kinetic energy to zero. The potential energy of the segment is

$$dV = \rho g y(x)\, ds$$

so the total (potential) energy is a functional of the path,

$$V = \rho g \int_0^d y\sqrt{1+y'^2}dx$$

We find the extrema of the total energy V. Rescaling, and varying in the usual way,

$$\begin{aligned}
0 &= \frac{1}{\rho g}\delta V \\
&= \int_0^d \left(\delta y\sqrt{1+y'^2} + \frac{1}{2}y\frac{2y'\delta y'}{\sqrt{1+y'^2}}\right)dx \\
&= \int_0^d \left(\delta y\sqrt{1+y'^2} + \frac{d}{dx}\left(\frac{yy'\delta y}{\sqrt{1+y'^2}}\right) - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right)\delta y\right)dx \\
&= \left.\frac{yy'\delta y}{\sqrt{1+y'^2}}\right|_0^d + \int_0^d \left(\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right)\right)\delta y dx \\
&= \int_0^d \left(\sqrt{1+y'^2} - \frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}}\right)\delta y dx
\end{aligned}$$

so that

$$\sqrt{1+y'^2} = \frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}}$$

To integrate, multiply by yy' and divide by $\sqrt{1+y'^2}$,

$$\begin{aligned}
yy' &= \frac{yy'}{\sqrt{1+y'^2}}\frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}} \\
y\frac{dy}{dx} &= \frac{yy'}{\sqrt{1+y'^2}}\frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}} \\
ydy &= \left(\frac{yy'}{\sqrt{1+y'^2}}\right)d\left(\frac{yy'}{\sqrt{1+y'^2}}\right) \\
\int ydy &= \int \left(\frac{yy'}{\sqrt{1+y'^2}}\right)d\left(\frac{yy'}{\sqrt{1+y'^2}}\right) \\
\frac{1}{2}y^2 - \frac{1}{2}y_0^2 &= \frac{1}{2}\left(\frac{yy'}{\sqrt{1+y'^2}}\right)^2 - \frac{1}{2}\left(\frac{y_0y_0'}{\sqrt{1+y_0'^2}}\right)^2
\end{aligned}$$

We know that the slope $y_0' = y'(0)$ vanishes because $x = 0$ is the lowest point of the hanging chain. Therefore,

$$\begin{aligned}
y^2 - y_{min}^2 &= \frac{y^2y'^2}{1+y'^2} \\
y^2 - y_{min}^2 &= \frac{y^2y'^2}{1+y'^2} \\
\left(y^2 - y_{min}^2\right)\left(1+y'^2\right) &= y^2y'^2 \\
y^2 - y_{min}^2 &= y^2y'^2 - \left(y^2 - y_{min}^2\right)y'^2 \\
y^2 - y_{min}^2 &= y_{min}^2y'^2
\end{aligned}$$

Solving for $y' = \frac{dy}{dx}$, allows us to integrate,

$$\begin{aligned}
\frac{dy}{dx} &= \pm\frac{\sqrt{y^2 - y_{min}^2}}{y_{min}} \\
\int_0^x dx &= \int_{y_{min}}^{y} \frac{y_{min} dy}{\sqrt{y^2 - y_{min}^2}} \\
x &= \int_{y_{min}}^{y} \frac{y_{min} dy}{\sqrt{y^2 - y_{min}^2}}
\end{aligned}$$

Now let $y = y_{min} \cosh \xi$ so that
Let

$$\begin{aligned}
y &= y_{min} \cosh \xi \\
dy &= y_{min} \sinh \xi d\xi
\end{aligned}$$

so that

$$\begin{aligned}
x &= \int_{y_{min}}^{y} \frac{y_{min} dy}{\sqrt{y^2 - y_{min}^2}} \\
&= \int_{y_{min}}^{y} \frac{y_{min}^2 \sinh \xi d\xi}{y_{min}\sqrt{\cosh^2 \xi - 1}} \\
&= y_{min} \int_{y_{min}}^{y} d\xi \\
&= y_{min}\xi \\
&= y_{min} \cosh^{-1} \frac{y}{y_{min}} - y_{min} \cosh^{-1} \frac{y_{min}}{y_{min}} \\
&= y_{min} \cosh^{-1} \frac{y}{y_{min}}
\end{aligned}$$

Now invert to solve for $y(x)$,

$$\begin{aligned}
x &= y_{min} \cosh^{-1} \frac{y}{y_{min}} \\
y(x) &= y_{min} \cosh \frac{x}{y_{min}}
\end{aligned}$$

Now h_1 and h_2 may be found by choosing x_1 and x_2 appropriately.