Two tricks

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1 Trick number one

Recall the variation of a general action of the form

$$S = \int L(q_i, \dot{q}_i, t)$$

Varying by computing $\left(S[q_i + \delta q_i] - S[q_i]\right)|_{linear\,order}$ where $\delta q_i(t)$ is arbitrary (it is the $h(t)$ we used before, but δq_i is more suggestive here) we find

$$\begin{aligned}
\delta S[q_1, q_2, \ldots, q_N] &= \left(S[q_i + h_i] - S[q_i]\right)|_{linear\,order} \\
&= \int L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) - \int L(q_i, \dot{q}_i, t)\Big|_{linear\,order}
\end{aligned}$$

Expanding the first integrand in a Taylor series to first order

$$L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t)_{linear\,order}| = L(q_i, \dot{q}_i, t) + \frac{\partial L}{\partial q_i}(q_i, \dot{q}_i, t)\,\delta q_i + \frac{\partial L}{\partial \dot{q}_i}(q_i, \dot{q}_i, t)\,\delta \dot{q}_i$$

Then the variation is

$$\begin{aligned}
\delta S[q_1, q_2, \ldots, q_N] &= \int \left(L(q_i, \dot{q}_i, t) + \frac{\partial L}{\partial q_i}(q_i, \dot{q}_i, t)\,\delta q_i + \frac{\partial L}{\partial \dot{q}_i}(q_i, \dot{q}_i, t)\,\delta \dot{q}_i\right) - \int L(q_i, \dot{q}_i, t) \\
&= \int \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right)
\end{aligned}$$

Note the similarity between this integrand and the total differential of L:

$$dL = \frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt$$

Because of this, all we need to do to find the variation is to think of it as taking the differential, with the simple replacements

$$\begin{aligned}
dq_i &\to \delta q_i \\
d\dot{q}_i &\to \frac{d}{dt}(\delta q_i) \\
dt &\to \delta t = 0
\end{aligned}$$

where δt vanishes because we only vary the path, not the time.

Example: Consider the first problem, where you are asked to vary

$$S = \int \left(\frac{1}{2}m\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{1}{2}K\left(ax^2 + 2bxy + cy^2\right)\right)dt$$

The differential of the integrand is

$$
\begin{aligned}
d\left(\frac{1}{2}m\left(a\dot{x}^2+2b\dot{x}\dot{y}+c\dot{y}^2\right)-\frac{1}{2}K\left(ax^2+2bxy+cy^2\right)\right) &= \frac{1}{2}m\left(2a\dot{x}d\dot{x}+2b\dot{y}d\dot{x}+2b\dot{x}d\dot{y}+2c\dot{y}d\dot{y}\right) \\
&\quad -\frac{1}{2}K\left(2axdx+2bydx+2bxdy+2cydy\right)
\end{aligned}
$$

so the variation of the action is

$$
\begin{aligned}
\delta S &= \int \frac{1}{2}m\left(2a\dot{x}\delta\dot{x}+2b\dot{y}\delta\dot{x}+2b\dot{x}\delta\dot{y}+2c\dot{y}\delta\dot{y}\right)dt \\
&\quad -\int \frac{1}{2}K\left(2ax\delta x+2by\delta x+2bx\delta y+2cy\delta y\right)dt
\end{aligned}
$$

2 Trick number two

The next step in the variation is to integrate the velocity variations by parts. This can be streamlined. For the first term in δS above:

$$
\begin{aligned}
\int_{t_1}^{t_2} ma\dot{x}\delta\dot{x} &= \int_{t_1}^{t_2} \frac{d}{dt}\left(ma\dot{x}\delta x\right)-\int_{t_1}^{t_2}\frac{d}{dt}\left(ma\dot{x}\right)\delta x \\
&= ma\dot{x}\left(t_2\right)\delta x\left(t_2\right)-ma\dot{x}\left(t_1\right)\delta x\left(t_1\right)-\int_{t_1}^{t_2}\frac{d}{dt}\left(ma\dot{x}\right)\delta x \\
&= -\int_{t_1}^{t_2} ma\ddot{x}\delta x
\end{aligned}
$$

The net result is simply the replacement under the integral sign,

$$
ma\dot{x}\delta\dot{x}\rightarrow -ma\ddot{x}\delta x
$$

that is, *throw the dot from the $\delta\dot{x}$ onto the rest of the expression and change the sign.*

If we use this trick, then after canceling the $\frac{1}{2}\times 2$ factorswe can immediately write δS as

$$
\begin{aligned}
\delta S &= \int m\left(a\ddot{x}\delta x+b\ddot{y}\delta x+b\ddot{x}\delta y+c\ddot{y}\delta y\right)dt \\
&\quad -\int K\left(ax\delta x+by\delta x+bx\delta y+cy\delta y\right)dt
\end{aligned}
$$

Now collect terms in δx and δy separately because the two variations are independent. We can hold x or y fixed while varying the other. We find

$$
\begin{aligned}
\delta S &= \int \left(m\left(a\ddot{x}+b\ddot{y}\right)-K\left(ax+by\right)\right)\delta x dt \\
&\quad +\int \left(m\left(b\ddot{x}+c\ddot{y}\right)-K\left(bx+cy\right)\right)\delta y dt
\end{aligned}
$$

and the equations of motion are

$$
\begin{aligned}
m\left(a\ddot{x}+b\ddot{y}\right)-K\left(ax+by\right) &= 0 \\
m\left(b\ddot{x}+c\ddot{y}\right)-K\left(bx+cy\right) &= 0
\end{aligned}
$$