Chapter 1

Review of Newtonian Mechanics

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3.1 Functionals

Informed discussion of Lagrangian methods is helped by introducing the idea of a *functional*. To understand it, think of a function, $f(x)$, as a mapping from the reals to the reals,

$$f : R \longrightarrow R$$

that is, given one real number, x, the functions hands us another real number, $f(x)$. This generalizes readily to functions of several variables, for example, $f(\mathbf{x})$ is a map from R^3 to R while the electric field $\mathbf{E}(\mathbf{x}, t)$ maps $E : R^4 \longrightarrow R^3$, since each choice of four coordinates (x, y, z, t) gives us three unique components of the electric field at that point.

In integral expressions, we meet a different sort of mapping. Consider

$$F[x(t)] = \int_{t_1}^{t_2} x(t)\, dt$$

where we introduce square brackets, [ ], to indicate that F is a *functional*. Given any function $x(t)$, the integral will give us a definite real number, but now we require the *entire function*, $x(t)$, to compute it. Define $\mathcal{F}$ to be a function space, in this case the set of all integrable functions $x(t)$ on the interval $[t_1, t_2]$. Then F is a mapping from this function space to the reals,

$$F : \mathcal{F} \longrightarrow R$$

Over the course of the twentieth century, functionals have played an increasingly important role. Introduced by P. J. Daniell in 1919, functionals were used by N. Weiner over the next two years to describe Brownian motion. Their real importance to physics emerged with R. Feynman's path integral formulation of quantum mechanics in 1948 based on Dirac's 1933 use of the Weiner integral.

We will be interested in one particular functional, called the *action* or *action functional*, given for the Newtonian mechanics of a single particle by

$$S[\mathbf{x}(t)] = \int_{t_1}^{t_2} L(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt$$

where the *Lagrangian*, $L(\mathbf{x}, \dot{\mathbf{x}}, t)$, is the difference between the kinetic and potential energies,

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = T - V$$

3.2 Some historical observations

At the time of the development of Lagrangian and Hamiltonian mechanics, and even into the 20^{th} century, the idea of a uniquely determined classical path was deeply entrenched in physicists' thinking about motion. The great deterministic power of the idea underlay the industrial age and explained the motions of planets. It is not surprising that the probabilistic preditions of quantum mechanics were strongly resisted[1] but experiment - the ultimate arbiter - decrees in favor of quantum mechanics.

This strong belief in determinism made it difficult to understand the variation of the path of motion required by the new approaches to classical mechanics. The idea of "varying a path" a little bit away from the classical solution simply seemed unphysical. The notion of a "virtual displacement" dodges the dilemma by insisting that the change in path is virtual, not real.

The situation is vastly different now. Mathematically, the development of functional calculus, including integration and differentiation of functionals, gives a language in which variations of a curve are an integral part. Physically, the path integral formulation of quantum mechanics tells us that one consistent way of understanding quantum mechanics is to think of the quantum system as evolving over all paths simultaneously, with a certain weighting applied to each and the classical path emerging as the expected average. Classical mechanics is then seen to emerge as this distribution of paths becomes sharply peaked around the classical path, and therefore the overwhelmingly most probable result of measurement.

Bearing these observations in mind, we will take the more modern route and ignore such notions as "virtual work". Instead, we seek the extremum of the action functional $S[\mathbf{x}(t)]$. Just as the extrema of a function $f(x)$ are given by the vanishing of its first derivative, $\frac{df}{dx} = 0$, we ask for the vanishing of the first *functional derivative*,

$$\frac{\delta S[x(t)]}{\delta x(t)} = 0$$

Then, just as the most probable value of a function is near where it changes most slowly, i.e., near extrema, the most probable path is the one giving the extremum of the action. In the classical limit, this is the *only* path the system can follow.

3.3 Variation of the action and the functional derivative

For classical mechanics, we do not need the formal definition of the functional derivative, which is given in Not so Classical Mechanics for anyone interested in the rigorous details. Instead, we make use of the extremum condition above and use our intuition about derivatives. The derivative of a function is given by

$$\frac{df}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

Notice that the limit removes all but the part of the numerator linear in ϵ, $f(x+\epsilon) - f(x) = \left(f(x) + \epsilon \frac{df}{dx} + \frac{1}{2}\epsilon^2 \frac{d^2 f}{dx^2} + \cdots \right) - f(x) \Rightarrow \left(f(x) + \epsilon \frac{df}{dx} \right) - f(x)$. The function at x cancels and we are left with the derivative. If the derivative vanishes, we do not need the dx part, but only

$$df = \left(f(x+dx) - f(x) \right)\big|_{linear\,order} = 0$$

where we have set $\epsilon = dx$. Applying the same logic to the vanishing functional derivative, we require

$$\delta S[x(t)] \equiv \left(S[x(t) + \delta x(t)] - S[x(t)] \right)\big|_{linear\,order} = 0$$

δS is called the *variation* of the action, and $\delta x(t)$ is an arbitrary variation of the path. Thus, if $x(t)$ is one path in the xt-plane, $x(t) + \delta x(t)$ is another path in the plane that differs slightly from the first. The

[1] Einstein's frustration is captured in his assertion to Cornelius Lanczos, "... dass [Herrgott] würfelt ... kann ich keinen Augenblick glauben." (I cannot believe for an instant that God plays dice [with the world]). He later abbreviated this in conversations with Niels Bohr, "Gott würfelt nicht...", God does not play dice. Bohr replied that it is not for us to say how God chooses to run the universe. See http://de.wikipedia.org/wiki/Gott_würfelt_nicht

variation δx is required to vanish at the endpoints, $\delta x(t_1) = \delta x(t_2) = 0$ so that the two paths both start and finish in the same place at the same time.

In defining the variation in this way, we avoid certain subtleties arising from places where the paths cross and $\delta x(t) = 0$, and also the formal need to allow δx to be completely arbitrary rather than always small. The variation is sufficient for our purpose.

Now consider the actual form of the variation when the action is given by

$$S[\mathbf{x}(t)] = \int_{t_1}^{t_2} L(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt$$

with $L(\mathbf{x}, \dot{\mathbf{x}}, t) = T - V$. For a single particle in a position-dependent potential $V(\mathbf{x})$, the action is given by

$$S[\mathbf{x}(t)] = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) dt$$

and setting $\delta\mathbf{x}(t) = \mathbf{h}(t)$, so we need to find $\left(S[\mathbf{x}+\mathbf{h}] - S[\mathbf{x}]\right)|_{linear\ order}$. Since we require both paths, $\mathbf{x}(t)$ and $\mathbf{x}(t) + \mathbf{h}(t)$, to go between the same endpoints at t_1 and t_2, we must have $\mathbf{h}(t_1) = \mathbf{h}(t_2) = 0$.

The vanishing variation gives

$$\begin{aligned}
0 &= \delta S[\mathbf{x}(t)] \\
&= \left(S[\mathbf{x}+\mathbf{h}] - S[\mathbf{x}]\right)|_{linear\ order} \\
&= \int_{t_1}^{t_2} dt \left(\left(\frac{1}{2}m\left(\dot{\mathbf{x}} + \dot{\mathbf{h}}\right)^2 - V(\mathbf{x}+\mathbf{h})\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right)\right)\Bigg|_{linear\ order} \\
&= \int_{t_1}^{t_2} dt \left(\left(\frac{1}{2}m\left(\dot{\mathbf{x}}^2 + 2\dot{\mathbf{x}}\cdot\dot{\mathbf{h}} + \dot{\mathbf{h}}^2\right) - V(\mathbf{x}+\mathbf{h})\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right)\right)\Bigg|_{linear\ order}
\end{aligned}$$

Now we drop the small quadratic term, $\dot{\mathbf{h}}^2$, cancel the kinetic energy $\frac{1}{2}m\dot{\mathbf{x}}^2$ along the original path $\mathbf{x}(t)$, and expand the potential in a Taylor series,

$$\begin{aligned}
0 &= \int_{t_1}^{t_2} dt \left(m\dot{\mathbf{x}}\cdot\dot{\mathbf{h}} - V(\mathbf{x}+\mathbf{h}) + V(\mathbf{x})\right)\Bigg|_{linear\ order} \\
&= \int_{t_1}^{t_2} dt \left(m\dot{\mathbf{x}}\cdot\dot{\mathbf{h}} - \left(V(\mathbf{x}) + \mathbf{h}\cdot\boldsymbol{\nabla}V(\mathbf{x}) + \mathcal{O}\left(\mathbf{h}^2\right) + \ldots\right) + V(\mathbf{x})\right)\Bigg|_{linear\ order} \\
&= \int_{t_1}^{t_2} dt \left(m\dot{\mathbf{x}}\cdot\dot{\mathbf{h}} - \mathbf{h}\cdot\boldsymbol{\nabla}V(\mathbf{x})\right)
\end{aligned}$$

Our next goal is to rearrange this so that only the arbitrary vector $\mathbf{h}$ appears as a linear factor, and not its derivative. We integrate by parts. Using the product rule to write

$$\frac{d}{dt}(m\dot{\mathbf{x}}\cdot\mathbf{h}) = m\ddot{\mathbf{x}}\cdot\mathbf{h} + m\dot{\mathbf{x}}\cdot\dot{\mathbf{h}}$$

and solving for the term, $m\dot{\mathbf{x}}\cdot\dot{\mathbf{h}}$, that we actually have, $m\dot{\mathbf{x}}\cdot\dot{\mathbf{h}} = \frac{d}{dt}(m\dot{\mathbf{x}}\cdot\mathbf{h}) - m\ddot{\mathbf{x}}\cdot\mathbf{h}$, the vanishing variation of the action implies

$$0 = \int_{t_1}^{t_2} dt \left(\frac{d}{dt}(m\dot{\mathbf{x}}\cdot\mathbf{h}) - m\ddot{\mathbf{x}}\cdot\mathbf{h} - \mathbf{h}\cdot\boldsymbol{\nabla}V(\mathbf{x})\right)$$

$$
\begin{aligned}
&= m\dot{\mathbf{x}}(t_2)\cdot\mathbf{h}(t_2) - m\dot{\mathbf{x}}(t_1)\cdot\mathbf{h}(t_1) - \int_{t_1}^{t_2} dt\,(m\ddot{\mathbf{x}} + \boldsymbol{\nabla}V(\mathbf{x}))\cdot\mathbf{h} \\
&= -\int_{t_1}^{t_2} dt\,(m\ddot{\mathbf{x}} + \boldsymbol{\nabla}V(\mathbf{x}))\cdot\mathbf{h}
\end{aligned}
$$

since $\mathbf{h}(t_2) = \mathbf{h}(t_1) = 0$.

Finally, suppose the integrand, except $\mathbf{h}(t)$ is nonvanishing at some point $\mathbf{x}(t')$. Then, since the integral must vanishing for all $\mathbf{h}(t)$, consider a choice of $\mathbf{h}$ parallel to the direction of $(m\ddot{\mathbf{x}} + \boldsymbol{\nabla}V(\mathbf{x}))$ and nonvanishing only in an infinitesimal region about $\mathbf{x}(t')$. Then the integral is approximately

$$
|m\ddot{\mathbf{x}}(t') + \boldsymbol{\nabla}V(\mathbf{x}(t'))|\,h(t') > 0
$$

This is a contradiction, so $m\ddot{\mathbf{x}}(t') + \boldsymbol{\nabla}V(\mathbf{x}(t')) = 0$. Since the point $\mathbf{x}(t')$ was arbitrary, the expression must vanish at every point by the same argument, and we have

$$
m\ddot{\mathbf{x}} = -\boldsymbol{\nabla}V(\mathbf{x})
$$

This is Newton's second law where the force is derived from the potential V.

3.4 The Euler-Lagrange equation

For many particle systems, we may write the action as a sum over all of the particles. However, there are vast simplifications that occur. For example, in a rigid body containing many times Avogadro's number of particles, the rigidity constraint reduces the number of degrees of freedom to just six - three to specify the position of the center of mass, and three more to specify the direction and magnitude of rotation about this center. More generally, the use of general coordinates and constraints may give expressions only vaguely reminiscent of the single particle kinetic and potential energies. Therefore, it is useful to take a general approach, supposing the Lagrangian to depend on N generalized coordinates q_i, their velocities, $\dot{q}_i$, and time. We take the potential to depend only on the positions, not the velocities or time, so that

$$
L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i)
$$

Despite the generality of this form, we may find the extrema of the action, which are the equations of motion in the coordinates q_i.

Carrying out the variation as before, the i^{th} position coordinate may change by an amount $h_i(t)$, which vanishes at t_1 and t_2. Following the same steps as for the single particle, vanishing variation gives,

$$
\begin{aligned}
0 &= \delta S[q_1, q_2, \ldots, q_N] \\
&= \left(S[q_i + h_i] - S[q_i]\right)\big|_{linear\,order} \\
&= \int_{t_1}^{t_2} dt \left(T\left(q_i + h_i, \dot{q}_i + \dot{h}_i, t\right) - V(q_i + h_i) - \left(T(q_i, \dot{q}_i, t) - V(q_i)\right)\right)\Bigg|_{linear\,order} \\
&= \int_{t_1}^{t_2} dt \left(\left(T(q_i, \dot{q}_i, t) + \sum_{i=1}^{N} h_i \frac{\partial T}{\partial q_i} + \sum_{i=1}^{N} \dot{h}_i \frac{\partial T}{\partial \dot{q}_i} - V(q_i) - \sum_{i=1}^{N} h_i \frac{\partial V}{\partial q_i}\right) - \left(T(q_i, \dot{q}_i, t) - V(q_i)\right)\right)\Bigg|_{linear\,order} \\
&= \sum_{i=1}^{N} \int_{t_1}^{t_2} dt \left(h_i \frac{\partial T}{\partial q_i} + \dot{h}_i \frac{\partial T}{\partial \dot{q}_i} - h_i \frac{\partial V}{\partial q_i}\right)
\end{aligned}
$$

where the Taylor series to first order of a function of more than one variable contains the linear term for each,

$$f(x+\epsilon, y+\delta) = f(x,y) + \epsilon\frac{\partial f}{\partial x} + \delta\frac{\partial f}{\partial y} + higher\ order\ terms$$

The center term contains the change in velocities, so we integrate by parts,

$$\begin{aligned}
\sum_{i=1}^{N}\int_{t_1}^{t_2} dt\,\dot{h}_i\frac{\partial T}{\partial \dot{q}_i} &= \sum_{i=1}^{N}\int_{t_1}^{t_2} dt\left(\frac{d}{dt}\left(h_i\frac{\partial T}{\partial \dot{q}_i}\right) - h_i\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\right) \\
&= \sum_{i=1}^{N}\left(h_i(t_2)\frac{\partial T}{\partial \dot{q}_i}(t_2) - h_i(t_1)\frac{\partial T}{\partial \dot{q}_i}(t_1)\right) - \sum_{i=1}^{N}\int_{t_1}^{t_2} dt\,h_i\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) \\
&= -\sum_{i=1}^{N}\int_{t_1}^{t_2} dt\,h_i\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)
\end{aligned}$$

The full expression is now,

$$\begin{aligned}
0 &= \sum_{i=1}^{N}\int_{t_1}^{t_2} dt\,h_i\left(\frac{\partial T}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i}\right) \\
&= \sum_{i=1}^{N}\int_{t_1}^{t_2} dt\,h_i\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)
\end{aligned}$$

where $L = T - V$ and we use $\frac{\partial V}{\partial \dot{q}_i} = 0$ to replace T with L in the velocity derivative term. Now, since each h_i is independent of the rest and arbitrary, each term in the sum must vanish separately. The result is the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{3.1}$$

3.5 General coordinate covariance of the Euler Lagrange equations

Here we show that the Euler-Lagrange equation is *covariant* under general coordinate transformations. By this we mean that if the Euler-Lagrange equation

$$V_i(x) \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

is satisfied in one set of coordinates, x^i, then it will hold in any other, y^i,

$$V_i(y) \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{y}^i} - \frac{\partial L}{\partial y^i} = 0$$

where $x^i\left(y^j\right)$ is the invertible coordinate transformation. For the two vectors to vanish together requires there to be a linear map from one to other, i.e., there exists some $J_i^{\ j}$ such that $V_i = \sum_j J_i^{\ j} V_j$, or

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i}\right) = \sum_{j=1}^{N} J_i^{\ j}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^j} - \frac{\partial L}{\partial y^j}\right)$$

It is clear what $J_i^{\ j}$ must be – if L is independent of velocity, we require

$$\frac{\partial L}{\partial x^i} = \sum_{j=1}^{N} J_i^{\ j}\frac{\partial L}{\partial y^j}$$

but the chain rule tells us that

$$\frac{\partial L}{\partial x^i} = \sum_{j=1}^{N} \frac{\partial y^j}{\partial x^i} \frac{\partial L}{\partial y^j}$$

Therefore, $J_i^{\ j}$ is the Jacobian matrix of the coordinate transformation, $\frac{\partial y^j}{\partial x^i}$. In conclusion, the Euler-Lagrangian equation hold in any coordinate system if and only if

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \sum_{j=1}^{N} \frac{\partial y^j}{\partial x^i} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^j} - \frac{\partial L}{\partial y^j} \right) \tag{3.2}$$

for any two, $x^i \to y^i$.

We prove that this is the case by deriving the relationship between the Euler-Lagrange equation for $x^i(t)$ and the Euler-Lagrange equation for $y^i(t)$.

Consider the variational equation for y^i, computed in two ways. Since the action may be written as either $S\left[x^i\right]$ or $S\left[y^i\right]$, we have

$$S\left[y^i\right] = S\left[x^i\left(y^k\right)\right]$$

First, we may immediately write the Euler-Lagrange equation by varying $S\left[y^i(t)\right]$. Following the steps that led us to Eq.(3.1), that is, varying and integrating by parts, we have

$$\begin{aligned}
\delta S &= \delta \int_C L\left(y^i, \dot{y}^i, t\right) dt \\
&= \sum_{k=1}^{N} \int_C \left(\frac{\partial L}{\partial y^k} \delta y^k + \frac{\partial L}{\partial \dot{y}^k} \delta \dot{y}^k \right) dt \\
&= \sum_{k=1}^{N} \int_C \left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k} \right) \right) \delta y^k dt
\end{aligned}$$

as expected. Now compare this to what we get by varying $S\left[x^i\left(y^k\right)\right]$ with respect to $y^i(t)$:

$$\begin{aligned}
0 &= \delta S \\
&= \delta \int_C L\left(x^i\left(y^k, t\right), \dot{x}^i\left(y^k, \dot{y}^k, t\right)\right) dt \\
&= \sum_{i,k=1}^{N} \int_C \left(\frac{\partial L}{\partial x^k}\left(\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial x^k}{\partial \dot{y}^i}\delta \dot{y}^i \right) + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{\partial \dot{x}^k}{\partial y^i}\delta y^i + \frac{\partial \dot{x}^k}{\partial \dot{y}^i}\delta \dot{y}^i \right) \right) dt
\end{aligned} \tag{3.3}$$

Since x^i is a function of y^k and t only, $\frac{\partial x^k}{\partial \dot{y}^i} = 0$ and the second term in the first parentheses vanishes.

Now we need two identities. Explicitly expanding the velocity, $\dot{x}^k$, the chain rule gives:

$$\begin{aligned}
\dot{x}^k &= \frac{dx^k}{dt} \\
&= \frac{d}{dt} x^k\left(y^i(t), t\right) \\
&= \frac{\partial x^k}{\partial y^i}\dot{y}^i + \frac{\partial x^k}{\partial t}
\end{aligned} \tag{3.4}$$

so differentiating, we have one identity,

$$\frac{\partial \dot{x}^k}{\partial \dot{y}^i} = \frac{\partial x^k}{\partial y^i}$$

For the second identity, we differentiate eq.(3.4) for the velocity with respect to y^i :

$$\begin{aligned}
\frac{\partial \dot{x}^k}{\partial y^i} &= \frac{\partial^2 x^k}{\partial y^i \partial y^j}\dot{y}^j + \frac{\partial^2 x^k}{\partial y^i \partial t} \\
&= \frac{\partial}{\partial y^j}\left(\frac{\partial x^k}{\partial y^i}\right)\dot{y}^j + \frac{\partial}{\partial t}\left(\frac{\partial x^k}{\partial y^i}\right) \\
&= \frac{d}{dt}\frac{\partial x^k}{\partial y^i}
\end{aligned}$$

Now return and substitute into the variation

$$\begin{aligned}
0 &= \delta S \\
&= \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k}\left(\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial x^k}{\partial \dot{y}^i}\delta \dot{y}^i\right) + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{\partial \dot{x}^k}{\partial y^i}\delta y^i + \frac{\partial \dot{x}^k}{\partial \dot{y}^i}\delta \dot{y}^i\right)\right) dt \\
&= \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d}{dt}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial x^k}{\partial y^i}\delta \dot{y}^i\right)\right) dt \\
&= \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d}{dt}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial x^k}{\partial y^i}\frac{d}{dt}\delta y^i\right)\right) dt \\
&= \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k}\frac{\partial x^k}{\partial y^i}\delta y^i + \frac{\partial L}{\partial \dot{x}^k}\frac{d}{dt}\left(\frac{\partial x^k}{\partial y^i}\delta y^i\right)\right) dt
\end{aligned}$$

Finally, integrate the final term by parts,

$$\begin{aligned}
\sum_{i,k=1}^{N}\int_C \frac{\partial L}{\partial \dot{x}^k}\frac{d}{dt}\left(\frac{\partial x^k}{\partial y^i}\delta y^i\right) dt &= \sum_{i,k=1}^{N}\int_C \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\frac{\partial x^k}{\partial y^i}\delta y^i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\frac{\partial x^k}{\partial y^i}\delta y^i\right) dt \\
&= \sum_{i,k=1}^{N}\left(\left(\frac{\partial L}{\partial \dot{x}^k}\frac{\partial x^k}{\partial y^i}\delta y^i\right)_{final} - \left(\frac{\partial L}{\partial \dot{x}^k}\frac{\partial x^k}{\partial y^i}\delta y^i\right)_{initial}\right) dt - \sum_{i,k=1}^{N}\int_C \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\frac{\partial x^k}{\partial y^i}\delta y^i dt \\
&= \sum_{i,k=1}^{N}\int_C \left(-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\frac{\partial x^k}{\partial y^i}\delta y^i\right) dt
\end{aligned}$$

where δy_i vanishes at the endpoints. The vanishing variation now becomes

$$0 = \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}\delta y^i dt$$

The initial equality of the two forms of the action, $S\left[y^i\right] = S\left[x^i\left(y^k\right)\right]$ implies $\delta S\left[y^i\right] = \delta S\left[x^i\left(y^k\right)\right]$ and therefore

$$\begin{aligned}
\sum_{k=1}^{N}\int_C \left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k}\right)\right)\delta y^k dt - \sum_{i,k=1}^{N}\int_C \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}\delta y^i dt &= 0 \\
\sum_{k=1}^{N}\int_C \left[\left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k}\right)\right) - \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}\right]\delta y^i dt &= 0
\end{aligned}$$

and the independence and arbitrariness of the variation, δy^i implies covariance:

$$\left(\frac{\partial L}{\partial y^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^k}\right)\right) = \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\frac{\partial x^k}{\partial y^i}$$

The conclusion we reach is that no matter what coordinates q^i we choose for a problem, we may *always* write the equation of motion as

$$\frac{\partial L}{\partial q^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right) = 0$$

The same is true of the action. Rather than writing the Euler-Lagrange equation, we may write the action as the integral of the Lagrangian and write the Lagrangian in terms of whatever coordinates we choose,

$$S\left[q^i\right] = \int_{t_1}^{t_2} L\left(q^i, \dot{q}^i, t\right) dt$$

Varying this with respect to the q^i will give the correct form of the equations.

3.6 Noether's Theorem

There are important general properties of Euler-Lagrange systems based on the symmetry of the Lagrangian. The most important symmetry result is Noether's Theorem, which we prove below. We then apply the theorem in several important special cases to find conservation of momentum, energy and angular momentum.

3.7 Noether's theorem for the Euler-Lagrange equation

Symmetries may be either discrete or continuous. Discrete symmetries like parity, time reversal, or the four rotations of a square, have only a finite number of possible transformations. By a continuous symmetry, we mean a symmetry dependent upon a real, continuous parameter such as a rotation through an angle θ, where θ may be any number between 0 and 2π.

In essence, Noether's theorem states that when an action has a continuous symmetry, we can derive a conserved quantity. To prove the theorem, we need clear definitions of a conserved quantity and of what we mean by a symmetry.

Def: Conserved quantities We have shown that the action

$$S\left[\mathbf{x}(t)\right] = \int_C L\left(x^i, \dot{x}^i, t\right) dt$$

is extremal when $x^i(t)$ satisfies the Euler-Lagrange equation,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0 \tag{3.5}$$

This condition guarantees that δS vanishes for *all* variations, $x^i(t) \to x^i(t) + \delta x^i(t)$ which vanish at the endpoints of the motion. Let $x^i(t)$ be a solution to the Euler-Lagrange equation, eq.(3.5) of motion. Then a function of $x^i(t)$ and its time derivatives,

$$f\left(x^i(t), \dot{x}^i(t)\ldots,\right)$$

is *conserved* if it is constant along the paths of motion,

$$\left.\frac{df}{dt}\right|_{x^i(t)} = 0$$

Definition: Symmetry of the action Sometimes it is the case that δS vanishes for certain limited variations of the path without imposing any condition at all. When this happens, we say that S has a *symmetry*:

A *symmetry* of an action functional $S[x]$ is a transformation of the path, $x^i(t) \rightarrow \lambda^i\left(x^j(t), t\right)$ that leaves the action invariant,

$$S\left[x^i(t)\right] = S\left[\lambda^i\left(x^j(t), t\right)\right]$$

regardless of the path of motion $x^i(t)$. In particular, when $\lambda^i(x)$ represents a continuous transformation of x[2], we may expand the transformation infinitesimally, so that

$$\begin{aligned} x^i &\rightarrow x'^i = x^i + \varepsilon^i(x) \\ \delta x^i &= x'^i - x^i = \varepsilon^i(x) \end{aligned}$$

Since the infinitesimal transformation must leave $S[x]$ invariant, we have

$$\delta_\varepsilon S = S\left[x^i + \varepsilon^i(x)\right] - S\left[x^i\right] = 0$$

whether $x(t)$ satisfies the field equations or not. If an infinitesimal transformation is a symmetry, we may apply arbitrarily many infinitesimal transformations to recover the invariance of S under finite transformations. Here $\lambda(x)$ is a *particular* function of the coordinates. This is quite different from performing a general variation – we are not placing any new demand on the action, just noticing that particular transformations do not change it. Notice that neither λ^i nor ε^i is required to vanish at the endpoints of the motion.

We are now in a position to prove Noether's theorem. Note that we carefully distinguish between the symmetry variation δ_ε and a general variation δ.

Theorem (Noether): Suppose an action dependent on N independent functions $x^i(t)$, $i = 1, 2, \ldots, N$ has a continuous symmetry so that it is invariant under

$$\delta_\varepsilon x^i = x'^i - x^i = \varepsilon^i(x)$$

where $\varepsilon^i(x)$ are fixed functions of $x^i(t)$. Then the quantity

$$I = \frac{\partial L(x(\lambda))}{\partial \dot{x}^i}\varepsilon^i(x)$$

is conserved.

Proof: The existence of a symmetry means that

$$\begin{aligned} 0 &\equiv \delta_\varepsilon S[x(t)] \\ &\equiv \sum_{i=1}^{N}\int_{t_1}^{t_2}\left(\frac{\partial L(x(t))}{\partial x^i}\varepsilon^i(x) + \left(\frac{\partial L(x(t))}{\partial \dot{x}^i_{(n)}}\right)\frac{d\varepsilon^i(x)}{dt}\right)dt \end{aligned}$$

Notice that $\delta_\varepsilon S$ vanishes identically because the action of δ_ε is a symmetry. No equation of motion has been used. Integrating the second term by parts we have

$$\begin{aligned} 0 &= \int\left(\frac{\partial L}{\partial x^i}\varepsilon^i(x) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\varepsilon^i(x)\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\varepsilon^i(x)\right)dt \\ &= \left.\frac{\partial L}{\partial \dot{x}^i}\varepsilon^i(x)\right|_{t_1}^{t_2} + \int\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right)\varepsilon^i(x)dt \end{aligned}$$

[2]Technically, what we mean here is a Lie group of transformations, but the definition of a group lines up well with our intuition of symmetry. Groups are sets closed under an operation which has an identity, inverses and is associative. For symmetries, each transformation leaves the action invariant, so the combination of any two does as well, showing closure. The identity is just no transformation at all, inverses are just undoing the transformation we've just done, and associativity is natural if you can picture it – compounding three transformations ABC it doesn't matter whether we find the effect of $D = AB$ and then find DC, or if we find $E = BC$ first then look at the effect of AE. It just means the symmetry transformation ABC is well-defined no matter which way we compute it, as long as we keep the original order.

This expression vanishes for *every* path. Now suppose $x^i(t)$ is a solution to the Euler-Lagrange equation of motion,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0$$

Then *along any such classical path* $x^i(t)$, the integrand vanishes and it follows that

$$\begin{aligned} 0 &= \delta S[\mathbf{x}] \\ &= \left.\frac{\partial L}{\partial \dot{x}^i}\varepsilon^i(x(t))\right|_{t_1}^{t_2} \\ &= I(t_2) - I(t_1) \end{aligned}$$

for any two end times, t_1, t_2. Therefore,

$$\frac{dI}{dt} = 0$$

and

$$I = \frac{\partial L(\mathbf{x}, \dot{\mathbf{x}})}{\partial \dot{x}^i}\varepsilon^i(x)$$

is a constant of the motion.

3.8 Examples of conserved quantities in Euler-Lagrange systems

3.8.1 Cyclic coordinates and conjugate momentum

We begin this section with some definitions.

Def: Cyclic coordinate A coordinate, q, is *cyclic* if it does not occur in the Lagrangian, i.e.,

$$\frac{\partial L}{\partial q} = 0$$

For example, in the spherically symmetric action

$$S[r, \theta, \varphi] = \int_{t_1}^{t_2}\left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) - V(r)\right]dt$$

all three velocities $\left(\dot{r}, \dot{\theta}, \dot{\varphi}\right)$ are present and the coordinates (r, θ) are present, but $\frac{\partial L}{\partial \varphi} = 0$. Therefore, φ is cyclic.

Def: Conjugate momentum The *conjugate momentum*, p, to any coordinate q is defined to be

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

For a single particle in any coordinate-dependent potential, $V(\mathbf{x})$, the action may be written as

$$S[\mathbf{x}] = \int_{t_1}^{t_2}\left[\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right]dt$$

so the momenta conjugate to the three coordinates x^i are

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i$$

reproducing the familiar expression for the momentum of a particle.

The conjugate momentum for a particle is not always simply $m\mathbf{v}$. If the particle moves in a velocity dependent potential, the form changes. The principal example of this is the Lorentz force law,

$$\mathbf{F} = q\left(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}\right)$$

which follows from the velocity-dependent potential

$$V\left(\mathbf{x}, \dot{\mathbf{x}}\right) = q\phi\left(\mathbf{x}\right) - q\dot{\mathbf{x}} \cdot \mathbf{A}\left(\mathbf{x}\right)$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Check this. The action, is $S\left[\mathbf{x}\right] = \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}} \cdot \mathbf{A}\right] dt$, so we see immediately differentiate to find the conjugate momentum

$$\pi_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i + qA_i$$

so that the momentum conjugate to x^i is

$$\boldsymbol{\pi} = m\dot{\mathbf{x}} + q\mathbf{A}$$

To check the equation of motion, we vary:

$$\begin{aligned}
0 &= \delta S\left[\mathbf{x}\right] \\
&= \int_{t_1}^{t_2} \left[m\dot{\mathbf{x}} \cdot \delta\dot{\mathbf{x}} - q\boldsymbol{\nabla}\phi \cdot \delta\mathbf{x} + q\delta\dot{\mathbf{x}} \cdot \mathbf{A} + q\dot{\mathbf{x}} \cdot \sum_i \frac{\partial \mathbf{A}}{\partial x^i}\delta x^i\right] dt \\
&= \int_{t_1}^{t_2} \left[-m\ddot{\mathbf{x}} \cdot \delta\mathbf{x} - q\boldsymbol{\nabla}\phi \cdot \delta\mathbf{x} - q\delta\mathbf{x} \cdot \dot{\mathbf{A}} + q\dot{\mathbf{x}} \cdot \sum_i \frac{\partial \mathbf{A}}{\partial x^i}\delta x^i\right] dt
\end{aligned}$$

where we have discarded the surface term from integrating the velocity variation by parts. Note that the final term contains a double sum, so we need the explicit summation rather than a second dot product. Then, extracting the variation δx^i, we expand the total time derivative of the vector potential. Since $\mathbf{A} = \mathbf{A}\left(x^i, t\right)$ with no velocity dependence, we have

$$\begin{aligned}
0 &= \sum_i \int_{t_1}^{t_2} \left[-m\ddot{x}_i - q\frac{\partial \phi}{\partial x^i} - q\frac{dA_i}{dt} + q\dot{\mathbf{x}} \cdot \frac{\partial \mathbf{A}}{\partial x^i}\right] \delta x^i dt \\
&= \sum_i \int_{t_1}^{t_2} \left[-m\ddot{x}_i - q\frac{\partial \phi}{\partial x^i} - q\left(\frac{\partial A_i}{\partial t} + \sum_j \frac{\partial A_i}{\partial x^j}\dot{x}^j\right) + q\sum_j \dot{x}^j\frac{\partial A_j}{\partial x^i}\right] \delta x^i dt
\end{aligned}$$

Now regroup,

$$0 = \sum_i \int_{t_1}^{t_2} \left[-m\ddot{x}_i - q\left(\frac{\partial \phi}{\partial x^i} + \frac{\partial A_i}{\partial t}\right) + q\sum_j \dot{x}^j\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)\right] \delta x^i dt$$

The term in brackets must vanish, and we recognize $\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)$ as the components of the curl and the final sum as the cross product of the curl with $\dot{x}^i$. Then

$$m\ddot{\mathbf{x}} = q\left(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}\right)$$

Exercise: Expand $\dot{\mathbf{x}} \times \mathbf{B}$ in terms of components of the velocity, where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ to show that the i^{th} component is

$$\left[\dot{\mathbf{x}} \times \mathbf{B}\right]_i = \sum_j \dot{x}^j\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)$$

3.8.2 Cyclic coordinates and conserved momentum

We have the following consequences of a cyclic coordinate.

Theorem: Cyclic coordinates If a coordinate q is cyclic then the momentum conjugate to q is conserved.

Proof: This follows immediately from Noether's theorem, since, if L is independent of q it is unchanged by replacing q with a translation $q' = q + a$ for any constant a. The action is therefore invariant under $\delta q = a$ and Noether's theorem gives the conserved quantity

$$\begin{aligned} I &= \frac{\partial L(\mathbf{x}, \dot{\mathbf{x}}, \dot{q})}{\partial \dot{q}} \varepsilon(x) \\ &= \frac{\partial L(\mathbf{x}, \dot{\mathbf{x}}, \dot{q})}{\partial \dot{q}} a \end{aligned}$$

But $p_q \equiv \frac{\partial L(\mathbf{x}, \dot{\mathbf{x}}, \dot{q})}{\partial \dot{q}}$ is the momentum conjugate to q and a is constant, so p_q is conserved.

3.8.3 Translational invariance and conservation of momentum

Now consider full translational invariance. We look first at a single particle, then at many particles.

Suppose the action for a 1-particle system is invariant under arbitrary finite translations,

$$\tilde{x}^i = x^i + a^i$$

or infinitesimally, letting $a^i \to \varepsilon^i$,

$$\delta x^i = \tilde{x}^i - x^i = \varepsilon^i$$

We may express the invariance of S under $\delta x^i = \varepsilon^i$ explicitly,

$$\begin{aligned} 0 &= \delta_{\varepsilon^i} S \\ &= \sum_i \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^i} \delta x^i + \frac{\partial L}{\partial \dot{x}^i} \delta \dot{x}^i \right) dt \\ &= \sum_i \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^i} \delta x^i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \delta x^i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \delta x^i \right) dt \\ &= \sum_i \frac{\partial L}{\partial \dot{x}^i} \varepsilon^i \bigg|_{t_1}^{t_2} + \sum_i \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \right) \varepsilon^i dt \end{aligned}$$

For a particle which satisfies the Euler-Lagrange equation, the final integral vanishes. Then, since t_1 and t_2 are arbitrary we must have

$$\frac{\partial L}{\partial \dot{x}^i} \varepsilon^i = p_i \varepsilon^i$$

conserved for all constants ε^i. Since ε^i is arbitrary, the momentum $p_i = \frac{\partial L}{\partial \dot{x}^i}$ conjugate to x^i is conserved as a result of translational invariance.

Now consider an isolated system, i.e., a bounded system with potentials depending only on the relavite positions, $\mathbf{x}_a - \mathbf{x}_b$ of the N particles $(a, b = 1, \ldots, N)$. We may write the action for this system as

$$S[\mathbf{x}] = \sum_{a=1}^{N} \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{\mathbf{x}}_a^2 - \sum_{b \neq a} V(\mathbf{x}_a - \mathbf{x}_b) \right) dt$$

Then shifting the entire system by the same vector $\mathbf{a}$,

$$\tilde{\mathbf{x}}_a = \mathbf{x}_a - \mathbf{a}$$

leaves S invariant since

$$
\begin{aligned}
\tilde{\mathbf{x}}_a - \tilde{\mathbf{x}}_b &= (\mathbf{x}_a - \mathbf{a}) - (\mathbf{x}_b - \mathbf{a}) = \mathbf{x}_a - \mathbf{x}_b \\
\dot{\tilde{\mathbf{x}}}_a &= \dot{\mathbf{x}}_a
\end{aligned}
$$

According to Noether's theorem, the conserved quantity is

$$
\begin{aligned}
I &= \frac{\partial L(\mathbf{x}, \dot{\mathbf{x}})}{\partial \dot{x}^i} \varepsilon^i(x) \\
&= \frac{\partial}{\partial \dot{x}^i} \sum_{a=1}^{N} \left(\frac{1}{2} m \dot{\mathbf{x}}_a^2 - \sum_{b \neq a} V(\mathbf{x}_a - \mathbf{x}_b) \right) a^i \\
&= \sum_{a=1}^{N} m \dot{x}_a^i a^i
\end{aligned}
$$

Finally, since a^i may be any constant vector we must have

$$
\mathbf{P} = \sum_{a=1}^{N} m \dot{\mathbf{x}}_a
$$

so that the total momentum is conserved for an isolated system.

3.8.4 Rotational symmetry and conservation of angular momentum (2 dim)

Consider a 2-dimensional system with free-particle Lagrangian

$$
L(x, y) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - V(\rho)
$$

where $\rho = \sqrt{x^2 + y^2}$ is the radial distance from the origin. Then rotation

$$
\begin{aligned}
x &\rightarrow x' = x \cos\theta - y \sin\theta \\
y &\rightarrow y' = x \sin\theta + y \cos\theta
\end{aligned}
$$

for any fixed value of θ leaves the action unchanged,

$$
S[\mathbf{x}] = \int L dt
$$

invariant. (Check this!)

For an infinitesimal change, $\theta \ll 1$, the changes in x, y are

$$
\begin{aligned}
\varepsilon^1 &= \delta x \\
&= x' - x \\
&= x \cos\theta - y \sin\theta - x \\
&= x \left(1 - \frac{1}{2!} \theta^2 + \ldots \right) - y \left(\theta - \frac{1}{3!} \theta^3 + \ldots \right) - x \\
&\approx -y\theta \\
\varepsilon^2 &= \delta y \\
&= y' - y \\
&\approx x\theta
\end{aligned}
$$

Therefore, from Noether's theorem, we have the conserved quantity,

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}^i}\varepsilon^i(x) &= m\dot{x}\varepsilon^1 + m\dot{y}\varepsilon^2 \\ &= m\dot{x}(-y\theta) + m\dot{y}(x\theta) \\ &= \theta m(x\dot{y} - y\dot{x})\end{aligned}$$

as long as x and y satisfy the equations of motion. Since θ is just an arbitrary constant to begin with,

$$\begin{aligned}J &= m(\dot{y}x - \dot{x}y) \\ &= xp_y - yp_x\end{aligned}$$

and we can identify the angular momentum,

$$\mathbf{J} = \mathbf{x} \times \mathbf{p}$$

as the conserved quantity.

It is worth noting that J is conjugate to a cyclic coordinate. If we rewrite the action in terms of polar coordinates, (r, φ), it becomes

$$S[r, \varphi] = \int_{t_1}^{t_2} \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$$

so that φ is cyclic. The momentum conjugate to φ is

$$\begin{aligned}p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} \\ &= mr^2\dot{\varphi}\end{aligned}$$

Differentiating $\tan\varphi = \frac{y}{x}$,

$$\begin{aligned}\frac{1}{\cos^2\varphi}\dot{\varphi} &= \frac{\dot{y}}{x} - \frac{y\dot{x}}{x^2} \\ \dot{\varphi} &= \frac{x\dot{y} - y\dot{x}}{x^2}\cos^2\varphi \\ &= \frac{x\dot{y} - y\dot{x}}{x^2}\left(\frac{x^2}{r^2}\right)\end{aligned}$$

giving the same result,

$$p_\varphi = mr^2\dot{\varphi} = m(x\dot{y} - y\dot{x}) = J$$

We will generalize this result to 3-dimensions after a complete discussion of rotations.

3.9 Conservation of energy

Conservation of energy is related to *time translation invariance*. However, this invariance is more subtle than simply replacing $t \to t + \tau$, which is simply a reparameterization of the action integral. Instead, the conservation law holds whenever the Lagrangian does not depend *explicitly* on time so that

$$\frac{\partial L}{\partial t} = 0$$

The total time derivative of L then reduces to

$$\begin{aligned}\frac{dL}{dt} &= \sum_i \frac{\partial L}{\partial x^i}\dot{x}^i + \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i + \frac{\partial L}{\partial t} \\ &= \sum_i \left(\frac{\partial L}{\partial x^i}\dot{x}^i + \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i\right)\end{aligned}$$

Using the Lagrange equations to replace

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}$$

in the first term, we get

$$\begin{aligned}
\frac{dL}{dt} &= \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i + \frac{\partial L}{\partial \dot{x}^i}\ddot{x}^i \right) \\
&= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i \right)
\end{aligned}$$

Bringing both terms to the same side, we have

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L \right) = 0$$

so that the quantity

$$E \equiv \sum_i \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L$$

is conserved. The quantity E is called the *energy*.

For a single particle in a potential $V(\mathbf{x})$, the conserved energy is

$$\begin{aligned}
E &\equiv \sum_i \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L \\
&= \sum_i \dot{x}^i \frac{\partial}{\partial \dot{x}^i}\left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) \\
&= \sum_i m\dot{x}^i\dot{x}^i - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) \\
&= \frac{1}{2}m\dot{\mathbf{x}}^2 + V(\mathbf{x})
\end{aligned}$$

For the velocity-dependent potential of the Lorentz force law,

$$S[\mathbf{x}] = \int_{t_1}^{t_2} \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right] dt$$

so that

$$\begin{aligned}
E &= \sum_i \dot{x}^i \frac{\partial}{\partial \dot{x}^i}\left(\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right) \\
&= \sum_i \dot{x}^i\left(m\dot{x}^i + qA^i\right) - \left(\frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}\right) \\
&= \frac{1}{2}m\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}}\cdot\mathbf{A} - (-q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}) \\
&= \frac{1}{2}m\dot{\mathbf{x}}^2 + q\phi
\end{aligned}$$

is conserved.

3.10 Scale Invariance

Physical measurements are always relative to our choice of unit. The resulting dilatational symmetry will be examined in detail when we study Hamiltonian dynamics. However, there are other forms of rescaling a problem that lead to physical results. These results typically depend on the fact that the Euler-Lagrange equation is unchanged by an overall constant, so that the actions

$$\begin{aligned} S &= \int L dt \\ S' &= \lambda \int L dt \end{aligned}$$

have the same extremal curves.

Now suppose we have a Lagrangian which depends on some constant parameters $(a_1, \ldots, a_n)$ in addition to the arbitrary coordinates,

$$L = L\left(x^i, \dot{x}^i, a_1, \ldots, a_n, t\right)$$

These parameters might include masses, lengths, spring constants and so on. Further, suppose that each of these variables may be rescaled by some factor in such a way that S changes by only an overall factor. That is, when we make the replacements

$$\begin{aligned} x^i &\rightarrow \alpha x^i \\ t &\rightarrow \beta t \\ \dot{x}^i &\rightarrow \frac{\alpha}{\beta}\dot{x}^i \\ a_i &\rightarrow \gamma_i a_i \end{aligned}$$

for certain constants $(\alpha, \beta, \gamma_1, \ldots, \gamma_n)$ we find that

$$L\left(\alpha x^i, \frac{\alpha}{\beta}\dot{x}^i, \gamma_1 a_1, \ldots, \gamma_n a_n, \beta t\right) = \lambda L\left(x^i, \dot{x}^i, a_1, \ldots, a_n, t\right)$$

for some constant λ which depends on the scaling constants. Then the Euler-Lagrange equations for the system described by $L\left(\alpha x^i, \frac{\alpha}{\beta}\dot{x}^i, \gamma_1 a_1, \ldots, \gamma_n a_n, \beta t\right)$ are the same as for the original Lagrangian, and we may make the replacements in the solution.

Consider the simple harmonic oscillator. The usual Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

If we rescale,

$$\begin{aligned} \tilde{x} &= \alpha x \\ \tilde{m} &= \beta m \\ \tilde{k} &= \gamma k \\ \tilde{t} &= \delta t \end{aligned}$$

then the rescaled Lagrangian is

$$\tilde{L} = \frac{1}{2}\frac{\beta\alpha^2}{\delta^2}m\dot{x}^2 - \frac{1}{2}\gamma\alpha^2 kx^2$$

and as long as $\frac{\beta}{\delta^2} = \gamma$, we have $\tilde{S} = \gamma\alpha^2 S$ as a scaling symmetry. Scaling x doesn't depend on the other scales, so there's no information there.

Now consider a system with unit mass and unit spring constant,

$$\begin{aligned} m_0 &= 1 \\ k_0 &= 1 \end{aligned}$$

and suppose this system is periodic, with period T_0. Then rescaling, the mass, spring constant and period become

$$\begin{aligned} m &= \beta m_0 = \beta \\ k &= \gamma k_0 = \gamma \\ T &= \delta T_0 \end{aligned}$$

and scale invariance tells us that a periodic solution also holds for the scaled m, k and T as long as $\delta = \sqrt{\frac{\beta}{\gamma}} = \sqrt{\frac{m}{k}}$. Therefore

$$T = T_0 \sqrt{\frac{m}{k}}$$

and the frequency is proportional to $\sqrt{\frac{k}{m}}$.